Attention Competition

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Zurich, April 06 2011

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Preface

"*Denn das ist eben die Eigenschaft der wahren Aufmerksamkeit, dass sie im Augenblick das Nichts zu Allem macht*" (Johann Wolfgang von Goethe, Wilhelm Meisters Wanderjahre)

This thesis establishes that attention is a very important, though ironically often neglected, element of economic theory. Receiving attention and being perceived is a fundamental requirement of any personal or corporate development as it stands at the beginning of all human interaction with others. In this respect I would like to devote my attention to those people who provided the essential support that this thesis could become reality. My special thanks go to my two supervisors, Josef Falkinger and Armin Schmutzler. Professor Falkinger accepted me as a member of his chair and unleashed my fascination for the field of limited attention. I am very grateful to him for numerous intriguing and illuminating discussions on general economic and non-economic topics. I wish to thank Professor Schmutzler for his many inspiring, motivating and supportive comments from the time where I was a graduate student up to the present. I also thank Diethard Klatte for his repeated and generous willingness to discuss all kind of mathematical problems and also Rainer Winkelmann with whom I had the pleasure to combine alpinism with economic reasoning. Moreover, I thank the members of the chairs of Armin Schmutzler and Josef Falkinger, especially my fellow students Sandra Hanslin, Timo Boppart, Donja Darai, Iryna Stewen and Victoria Galsband for many valuable discussions and pleasurable moments. Andreas Haller and Silvie Gernet provided excellent research assistance. I also would like to express my gratitude to my father, who inspired me to search for my own answers and my mother, who gave me the enthusiasm not to give up in such a struggle. Finally, I express my unlimited thanks to my fiancee, Ines Brunner,

for her endless support during harder times, her love, her kindness, her patience and her faith in me.

Andreas M. Hefti

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Introduction

1.1 Superabundant information and scarce attention

In a seminal paper Herbert Simon revises the notion of the "rational economic man" by introducing "internal constraints" which account for the fact that besides the conventional budget constraint that represents economic scarcity an agent also faces physiological and psychological limitations to which he must obey (Simon (1955)). Consecutively, much theoretical and empirical effort was undertaken to comprehend what the admission of such bounds might imply for economic models (see Conlisk (1996) for a survey). In many cases bounds on the information processing capabilities of agents have been found of central importance. The claim of these models usually is that acquiring and processing information imposes some cost on the agent. The existence of such deliberation costs then leads to a trade-off in decision-making (see Payne et al. (1993)): better decisions are available only at the expense of purchasing more information.

However, the information problem in a modern economy might be less one of getting information but one of receiving too much information. Maybe the central phenomen of the digital age is that "Information has gone from scarce to superabundant[1]". One reason for this abundance can be seen in advertising efforts of firms. The following two figures illustrate real annual advertising expenditure for the U.S. from 1980 - 2008.[2]

[1]Cited in: The Economist, "Data, data everywhere. A special report on managing information" Feb 27th 2010, p.3.

[2]Source: Magna Global (published by Television Bureau of Advertising: http://www.tvb.org/nav/buildframeset.aspx).

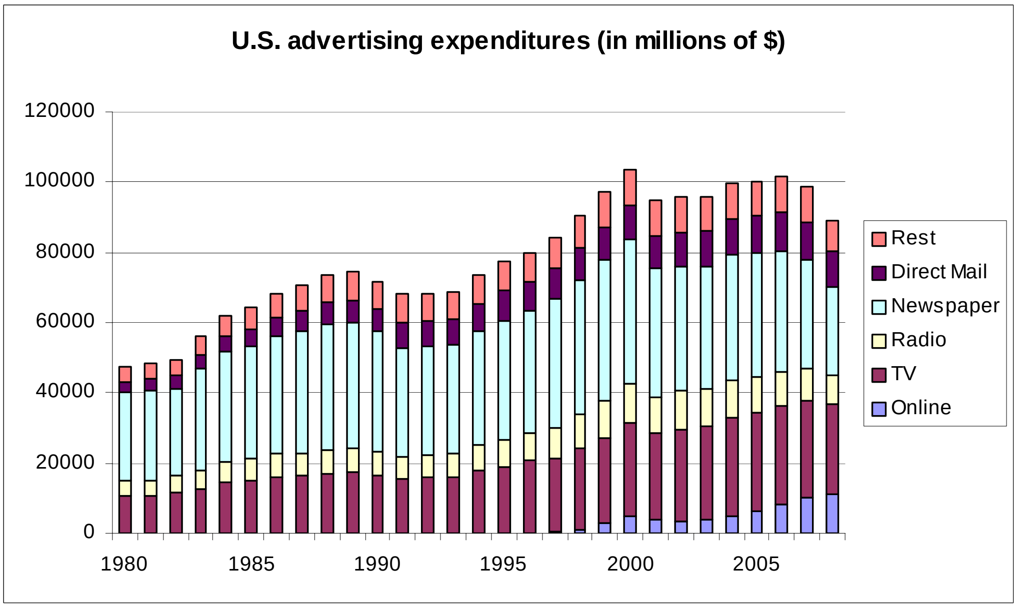

Figure 1.1: Real annual advertising expenditure (U.S.): 1980-2008

We see that overall real advertising expenditure has increased over time. Further we note that the composition of advertising expenditures remains constant over time with two important exceptions. First, we see that the share of expenditures on newspaper advertising tends to diminish over time. Second, we see that the share spent on online advertising rapidly increases after its introduction around 1996. The figure suggests that after 1996 online advertising may be responsible for the relative decline of the share spent on newspaper advertising. Figure 1.3 shows the decomposition of online advertising expenditures: We see that initially the expenditures on online advertising were dominated by on site advertising. In more recent years this dominance has been overthrown by the expenditure for paid search. To get some sense of the magnitude of advertising expenditures figure 1.4 depicts the ratio of U.S. advertising expentitures to consumption expenditures[3]. We see that this share is rather constant around 1% until mid seventies but then increases steadily for about two decades and stabilises around 3%.

As a general fact we see that real advertising expenditure in the U.S. has more than doubled in the last thirty years and that the slope of the advertising-consumption share changes remarkably around 1975. Online advertising constitutes a comparably small share

[3]Source: Consumption data stems from the Penn World Table 6.3. Advertising expenditure stems from the Historical Statistics of the United States database.

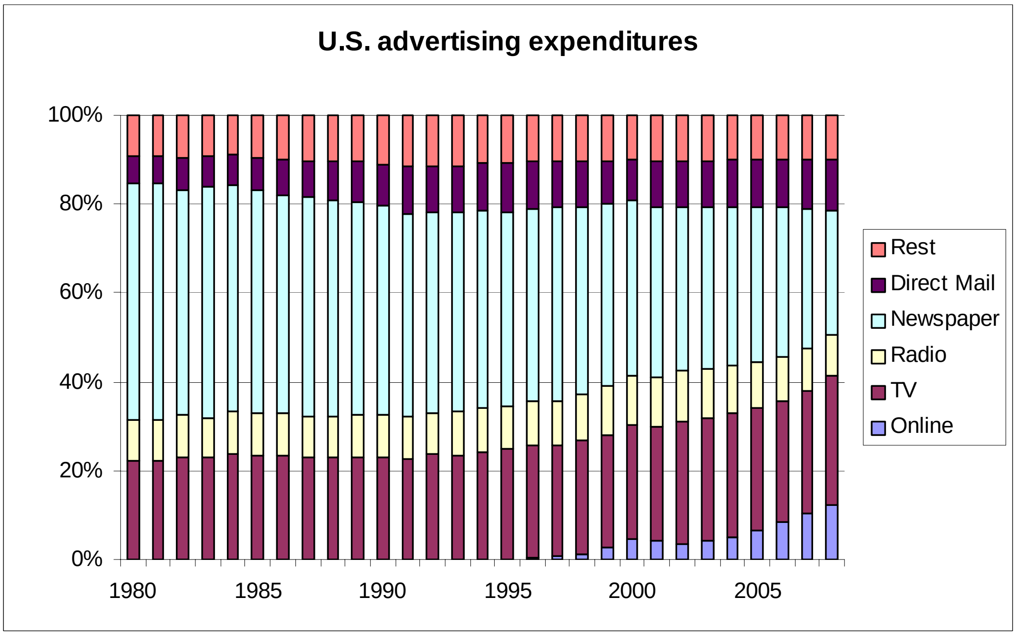

Figure 1.2: Advertising expenditure (U.S.): Decomposition

of overall advertising expenditure at the moment but appears to become more and more important which appears to be driven mostly by the expenditures on paid search.

But the increase of advertising in the last thirty years is only one example of the glut of information that characterises a modern economy. The virtual explosion of information flow is well documented by the omnipresence of the internet with its ever-increasing number of web sites. The next figure shows that the fraction of U.S. internet users increased more than four times during the last 15 years.[4] But also more traditional sources of information have become a lot more voluminous. For example, Davenport and Beck (2001, p.4) report that the Sunday New York Times contains more factual information in one edition than was available to a reader in all the written material in the fifteenth century.

Recent work in psychology highlights the importance of limitations on perceiving multiple stimuli for making decisions, storing information, planning actions and other mental processes (Pashler (1998)). Together with such limitations the modern "wealth of information" then implies a "scarcity of attention" (Simon (1971)). Managing scarce attention is seen to be the central task of modern business: "If you want to be successful in the cur-

[4]Source: World Development Indicators Database.

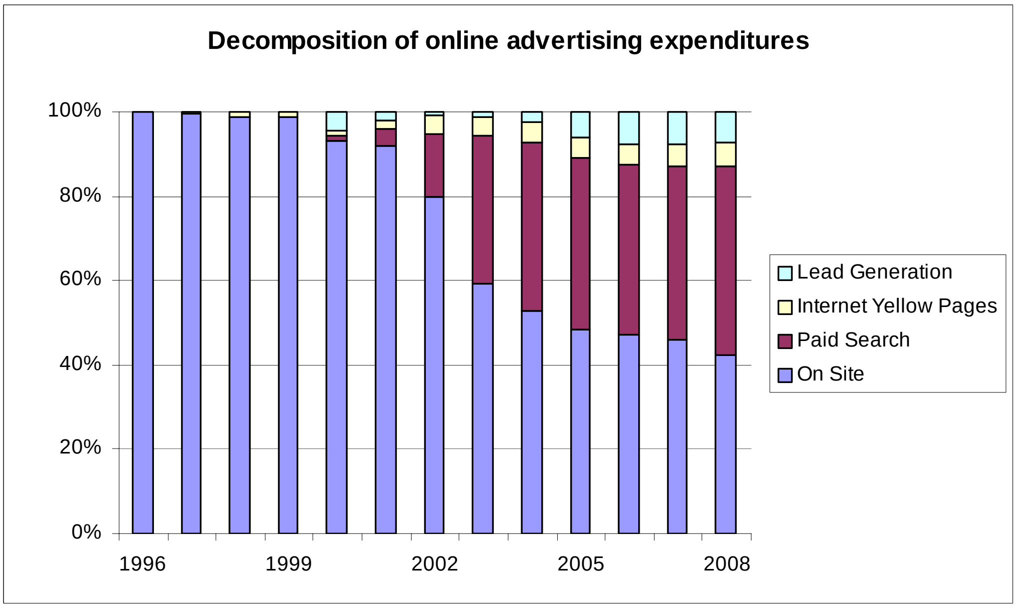

Figure 1.3: Online advertising expenditures (U.S.)

rent economy, you've got to be good at getting attention" (Davenport and Beck (2001), p. 8). What are the consequences of scarce attention for economic theory? It is the main task of this thesis to theoretically investigate the implications of limited consumer attention in the context of oligopolistic theory of competition.

1.2 Stylized facts from psychology and marketing science

In this section I discuss the relevant literature on attention from perceptual psychology, marketing research and economics and work out the main stylized facts which will serve as building blocks for the later chapters. One central question of psychological work on attention is "whether attention is goal-driven, controlled in a top-down fashion, or stimulus driven, controlled in a bottom-up fashion" (Yantis (1998), p. 223). This distinction of attention as an active or passive part of perception is also reflected in the dichotomy of the modern economic literature on attention where a vast part of the literature is concerned with goal-driven rather than stimulus-driven allocation of attention. For example, in models of rational inattention (Reis (2006a) and Reis (2006b), also Gifford (2005)) information

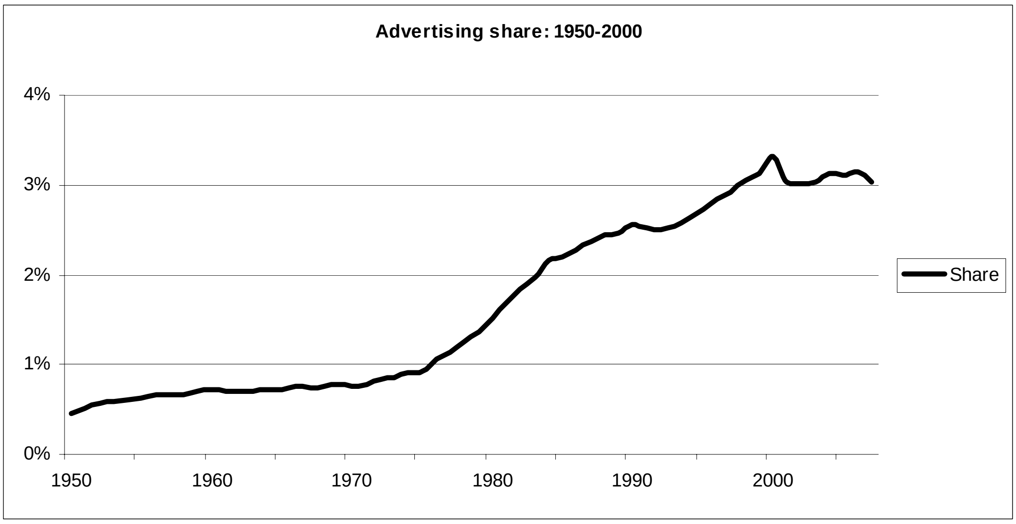

Figure 1.4: Advertising share of consumption expenditure (U.S.)

acquisition is costly and a part of the agent's maximization problem. In the dynamic context of these models the consumers only sporadically update their information which is interpreted as an agent being inattentive for some time concerning further information. Sims investigates the implications of rational inattention on macroeconomic topics such as the permanent income hypothesis using Shannon's information theory (Sims (2003)). Gabaix and Laibson develop a cost-benefit model that endogenously explains how subjects allocate their limited mental resources to different elements of a decision problem and provide experimental evidence in support of their directed cognition effect (Gabaix et al. (2003) and Gabaix et al. (2006)). In finance limited attention of market actors has been taken into account by different researchers. For example, Mondria et al. show that the magnitude of home bias in investments can be explained substantially by the allocation of investor's limited attention (Mondria et al. (2010)). In their model investors use their limited mental capacity to process information about the future payoff of risky assets. As capacity limitations impose an upper bound on the possibility of the investors to reduce their uncertainty the investors have to decide which assets from which countries deserve closer attention. Using a three-month AOL dataset from 2006 that contains all web search queries of the AOL users these authors show that i) investors allocate more attention (as measured by search queries and clicks) to countries of which they already hold assets and ii) investors favor assets from already familiar countries. Gifford

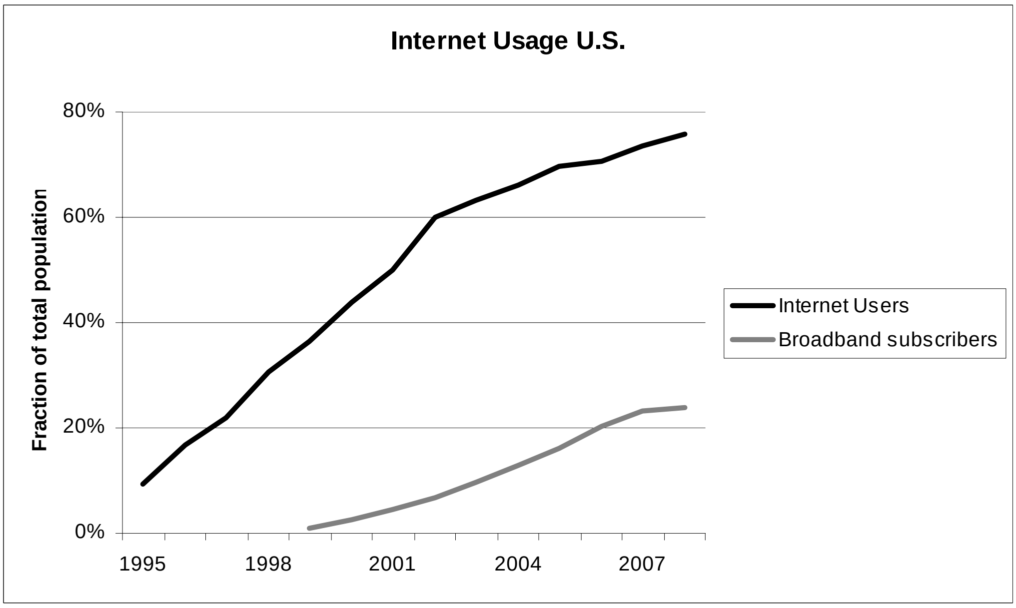

Figure 1.5: Fraction of internet and broadband subscribes in the U.S.

presents a dynamic principal-agent model of venture capital where the venture capitalist (the agent) must decide in each period how much attention (or time) to allocate towards managing a current venture or towards funding new ventures (Gifford (1997)). As the venture capitalists' allocation problem is not taken into account by an entrepreneur (the principal) the venture capitalist allocates too little time towards managing the venture from the perspective of the entrepreneur but also from the perspective of limited venture partners. Using an NYSE dataset from 2002 Corwin and Coughenour show that the ability of specialist traders to provide liquidity to a stock is negatively correlated with the attention required by the other stocks in the portfolio (Corwin and Coughenour (2008)). Peng shows that investors allocate their limited attention across sources of uncertainty to minimize total portfolio uncertainty (Peng (2005)), and Huberman (Huberman (2001)) as well as Barber and Odean (Barber and Odean (2008)) provide evidence that investors tend to focus on familiar stocks and that information may not be incorporated into prices until it attracts sufficient investor attention.

The smaller part of the economic literature on attention focuses on stimulus-driven attention allocation. If attention is stimulus-driven this naturally moves the behaviour of the information sender into the centre of interest. As the main example of this literature Falkinger (Falkinger (2007), Falkinger (2008)), derives two macroeconomic models where consumer attention is not a decision variable and attention is being allocated passively by the strength of the signals a consumer receives. In his view attention allocation only matters if there is an abundance of information in the economy (what he calls the information-rich economy) so that the limited mental capacity of consumers becomes a scarce resource. Then the relative salience of its messages are the main information problem of a firm (the information sender) as salience determines whether a message among many other messages is perceived or not. In the information-rich economy an economic interaction between a sender and a receiver previously requires that the attention of the receiver has been attracted by the sender, i.e. that the sender wins a competition for attention before competing for economic resources. From the viewpoint of a firm being on an attention-limited consumer's mind is important for two reasons: 1) the consumer is informed of its product and 2) she also is not (or at least less) thinking about rival products. On the contrary, if there only is little information in the economy (the economy is information-poor) then information and not attention is scarce and relative salience of a message is far less important than whether the message is actually received by a consumer or not. Falkinger shows that whether an economy is information-rich or information poor is determined by fundamentals such as preferences, budget, information technology and production technology. Further, it is shown that in an information-rich world international integration tends to reduce global diversity (as measured by the total number of items an impartial observer would count) and so does media intermediation. Hirshleifer and Teoh construct a model of security trading where some investors are inattentive which in their model means that these investors only consider a subset of all publicly available information whereas attentive investors consider the entire information (Hirshleifer and Teoh (2003)). They show that equilibrium prices reflect the different believes of attentive and inattentive investors in a simple model of security trading. Further, these authors illustrate the implications of limited investor attention on stock prices if the manager of a firm can influence the perception of a financial report by disclosing additional material. They show that managers have an incentive to include additional pro forma (non-GAAP) measures on earnings to improve the perception of the firm as inattentive investors treat pro forma earnings as if they were adjusted by the manager to be maximally informative. This failure of the inattentive investors to account for the strategic incentive of the manager implies that i) pro forma earnings are upward biased predictions of terminal cash flows and ii) stock prices are too high compared to a situation were reporting of pro forma earnings were forbidden. Compared to the case where all investors are attentive the existence of inattentive investors implies average overvaluation of a stock and hence also more negative average stock return should the abnormal (or bad) state occur.

The modern literature on experimental psychology provides evidence for both the active and the passive role of attention in the context of visual stimulus processing[5]: "goal-driven" and "stimulus driven" attention are important in explaining how subjects perform during visual search experiments. Nevertheless "stimulus-driven attentional control is both faster and more potent than goal-driven attentional control" (Yantis (1998), p. 251-252) - especially in experiments where attention cannot be focused on a certain region in advance. This literature argues that attention operates as a gating mechanism if multiple stimuli use the same neural pathway by "restricting the amount of information that is processed at once" (Mozer and Sitton (1998), p. 342) because parallel processing of information is not possible due to mental capacity limitations. Then the relative strength of a stimulus that enters the neural network of a person determines whether or not it is further transmitted into the neural network and eventually reaches the recognition network (Mozer and Sitton (1998), p. 343-356). That capacity limitations impose a "bottleneck" to visual stimulus processing is convincingly illustrated in experiments on visual search: response time of subjects to a stimulus is flat if few objects are displayed but increases exponentially with the number of displayed objects (Mozer and Sitton (1998), p. 378-340). Hence perception gets distracted under many signals which ultimately slows down response time. In explaining search time in visual search the "relative salience of a target" is much more important than "the occurrence of specific features" (Nothdurft (2000), p. 1184). Hence in order to generate a "pop-up" or "salience effect" the motion, color or luminance of an object matters only relatively to the local and global surrounding of the object. Experiments conducted to assess the spatial distribution of attention on the visual field suggest that attention works as a "spotlight": once a location is selected, all features at that location are processed and moved on to the recognition network (e.g.

[5]In such experiments subjects are usually required to identify the orientation of some geometric object that e.g. briefly flashes up where there might also be distractive stimuli present and response time of subjects is measured. See Yantis (Yantis (1998)) for an overview of such experiments.

Kahneman and Henik (1981)) or at least "receive enhanced processing" (Maunsell and Treue (2006)). We may summarise the key messages of the psychological research in case of visual attention as follows.

P1) In the case of multiple stimuli relative signal strength matters whether a signal is processed or not.

P2) For a given spatial array of objects, if a region successfully attracts attention then all features of this region are considered.

Note that P1) also means that in case of multiple overlapping stimuli certain signals are inhibited[6] and not processed to the recognition network.

While the general economic model I develop in chapter 3 of this thesis is in principal capable of handling both cases - active (or "goal-driven") and passive (or "stimulus-driven") attention allocation, I will follow the Falkinger-Simon approach and consider the case where consumer attention is passive in its nature and the distribution of attention will depend on the distribution of signal strength.

In the marketing literature it is a well-documented fact that the placement of a link on the screen of some online search site is a crucial determinant of the number of clicks that a site receives. For example using a dataset of a price listing service Baye et al. find that "a firm receives about 17% fewer clicks for every competitor listed above it on the screen" (Baye et al. (2009), p. 938) but that the relationship between clicks and screen location is far from being linear. That ranking matters for clicks on search pages is also confirmed by Smith and Brynjolfsson (Smith and Brynjolfsson (2004)) in the context of online book markets and by Ghose and Yang in the context of sponsored search advertising (Ghose and Yang (2009)). Having sales data in case of memory modules Ellison and Ellison (Ellison and Ellison (2009), p. 442) report that on a price search engine "moving from first to seventh on the list reduces a websites sales [...] by 83%". Dreze and Zufryden, using data on the web traffic of Amazon and Ebay, point out that a site can increase its visibility to potential consumers most by altering its linkage in the web and thereby improving its indexing by search engines (Dreze and Zufryden (2004), p. 35). Most noteworthy is the

[6]See Milliken and Tipper (1998), p. 204-206 for experiments on the importance of inhibition in selection experiments.

finding that the first three positions get about 75% out of all clicks of the first ten positions on the screen (Baye et al. (2009), p. 945) and positions with rank larger than three do not differ substantially from each other. This finding is also confirmed by Ghose and Yang who additionally document a non-monotonic relationship between rank and profitability (Ghose and Yang (2009), p. 1614). This is explained by the fact that both costs and clicks decrease with a higher rank (a lower on-screen position) but that clicks decrease slower than costs. Hence it may be of less importance to be the first on the list but what matters is to be among the first few entries. A related finding of Smith and Brynjolfsson is that in case of the market for books being on the first page of a search page is of central relevance for receiving a high number of clicks and might even be more important than being the overall first in the list (Smith and Brynjolfsson (2004), p. 548-549). First-page dominance is also found by Jansen et al. (Jansen et al. (2000), p. 215). Finally, Pan et al. show in an eyeball-tracking experiment that the first entries on a Google search page get by far most viewing-time but there is no statistical difference between the first and the second item on the list (Pan et al. (2007), p. 814). In their online experiment Pan et al. further show that people frequently select the upper items of a list in a quality based search experiment even if these upper items systematically are of inferior quality. Hence the Pan finding suggests that people consider the first few alternatives in making their decisions *independent* of the overall quality of these alternatives.

The literature on marketing has recognised the empirical fact that consumers make their decision conditional on a subset of all possible alternatives. In such models consumers are believed to phase their decision in two parts using simple heuristics to screen all possible alternatives and then make a more thorough analysis of the reduced consideration set (Manrai and Andrews (1998); see Payne et al. (1993), Chapter 2, for an overview of different decision strategies). Hauser and Wernerfeld (Hauser and Wernerfelt (1990)) provide an overview of marketing studies that find empirical evidence for the existence of such consideration sets and emphasise that these sets usually are very small (a size of $3-6$ alternatives) compared to the complete set of possible alternatives[7]. Moreover, the consideration sets of supermarket shoppers seem to depend heavily on what is promoted

[7] In case of shampoos the mean consideration set contained only four shampoos out of more than 30 (Hauser and Wernerfelt (1990)).

strongly (Fader and McAlister (1990), p. 330-331). This is also supported by Mitra who reports that advertising does not alter the average size of the consideration set but it does influence which alternatives are in the consideration set in an experimental setting (Mitra (1995), p. 91). Mehta et al. use a structural model of quality-adjusted price search to estimate in-shop brand selection in case of liquid detergents and find that consumers frequently purchase low quality products "simply because they fail to notice the prices of the other brands" (Mehta et al. (2003), p. 76) and, at the same time, it is "the most feature-advertised and displayed brand". Similarly, Alba and Chattopadhyay find experimental evidence that higher salience of a brand inhibits the recall of other brands (Alba and Chattopadhyay (1986), p. 365). The key messages of marketing and internet research are:

M1) The on-screen ranking of links determines the clicking distribution and is sale relevant.

M2) There appears to be a discontinuity and non-linearity between the clicks of the first few entries and the later entries of search pages and there is first-page relevance.

M3) Consideration sets of consumers exist and are small. The size of the set is not affected e.g. by advertising. Firms can influence their chance of consideration e.g. by means of promotion.

Note that P1) and P2) can explain M1) in the sense that people cannot process all information on a search screen in parallel and thus perceive the links from top to bottom. That flashy links by means of position or different color attract attention and hence clicks (the finding of Ghose and Yang (2009)) coincides with Nothdurft's (2000) observation that the relative salience of objects determines the speed of their detection and also is reflected by the fact that sponsored search advertising (purchasing advertising space and a different link color in the Google top- and sideframe) has become "the largest source of revenues for search engines" (Ghose and Yang (2009), p. 1605, also see figure 1.3). Further P2) and M2) seem related as the empirical evidence suggests the clicking behavior to cluster around the first search page and the first entries of a page. Finally, M2) and M3) are related as the discontinuity could be explained by the fact that people consider only a small subset of all information and their selection rule follows a simple top-down logic.

1.3 Outline

The facts documented in the last section suggest that:

1. Decision-makers have limited attention: They focus only on a subset of all available information.

2. Relative salience of the information (e.g. its on-screen ranking) crucially determines which information receives attention.

These findings are incorporated by Falkinger in a general equilibrium framework with a clear macroeconomic thrust (Falkinger (2007) and Falkinger (2008)). In his models firms have no mass and hence no strategic interaction exists. A natural alternative to his approach is to integrate limited attention into a model of oligopolistic competition that allows for strategic interactions of information senders. My thesis presents an attempt to accomplish this goal by developing a game-theoretic setting that combines attention competition with economic price competition. One advantage of my approach is that we can investigate how the competition for limited attention affects the strategic choice of prices and attention efforts of the competing firms. That is, we can understand in greater detail how the market forces under limited attention eventually establish an equilibrium. At the positive level I show in chapter 3 that under reasonable assumptions a strategic equilibrium exists in the symmetric price-attention game and discuss how limited attention affects market structure, attention efforts and prices. I provide some examples of standard models of oligopolistic price competition that satisfy these assumptions in chapter 4 and use one of them, the Salop-Grossman-Shapiro model of circular product differentiation (Salop (1979), Grossman and Shapiro (1984)), to discuss the consequences of attention competition at the normative level. I now present the outline of my thesis in greater detail.

The second chapter of this book integrates the first finding - consumers have limited attention - into the Grossman and Shapiro model of informative advertising and circular product differentiation (Grossman and Shapiro (1984)) by assuming that consumers are only capable of considering a certain subset of all alternatives they are informed of. In this model firms and consumers are located symmetrically around the unit circle and the overall number of firms may be interpreted as a measure of diversity: more firms mean more diversity. The basic finding is that perceived diversity and not effective diversity (as measured by the total number of firms) matters for the economic equilibrium, which leaves firms with considerably more market power and implies that higher equilibrium prices and a higher degree of diversity can be sustained than predicted by the conventional model of unlimited attention capacities.

In the third chapter I combine the second observation - that ranking matters - with an abstract model of price competition. The possibility of information-senders to influence their chance of perception triggers a competition for consumer attention which, depending on the severity of this competition, may increase or decrease equilibrium profits compared to the solution under the special assumption of unbounded attention. In this chapter I also develop a concrete and intuitive functional form of the attention competition - the attention contest function - which because of its analytically nice properties can easily be implemented in symmetric models of economic competition from oligopoly theory. The third chapter has a prolonged technical section in which I derive conditions that assert existence and uniqueness of the symmetric price-attention game. The game-theoretic tools required are developed in the fifth chapter of this thesis.

The fourth chapter of this book first embeds attention competition as developed in the third chapter in a standard model of price competition from oligopoly theory. I show that this model of simultaneous price-attention competition satisfies the main general prerequisites from the third chapter that assert existence and uniqueness of a price-attention equilibrium. Then I use this model to discuss the issue of asymmetric access to attention technology. The main and also worrisome finding is that firms with substantial advantages in the possibility to attract attention (e.g. because of a previous investment into marketing research) can crowd out more efficient competitors in terms of production costs that fail to get sufficient attention. Then I use the circular model of Salop (Salop (1979)) and include attention competition as developed in chapter 3. One important feature of the Salop model is that it allows to discuss the welfare implications of attention competition because the distance between the location of the consumer and the location of the firm determines the utility a consumer experiences by purchasing at the location of the firm. The further consumer and firm are apart, the higher are transportation costs burdened on the consumer and the lower is his utility from the transaction. Without limited attention there is a negative relationship between transportation costs and diversity (as measured by the total number of firms on the circle). More diversity decreases average transportation costs because if more firms are located symmetrically around the unit circle the distance between the firms then must decrease. But under unlimited attention consumers may perceive the entire market and hence can benefit from the firms moving closer together. A very intriguing finding is that limited attention reverses the conventional negative relationship between transportation costs and diversity which is also worrisome as attention competition implies equilibrium diversity to be a lot larger than under the conventional solution. It then comes only at little surprise that under limited attention a planning authority would primarily focus on cutting back extensive diversity. The last part of chapter four combines attention competition in the circular model with informative advertising. Using numerical methods I show the basic results on transportation costs and inefficiency of the attention equilibrium also to be true in this more general case.

As mentioned before the fifth chapter of this book is methodological and deals with uniqueness and stability of general differentiable symmetric games. In this chapter I derive two criteria that, if both are satisfied, imply uniqueness of the symmetric equilibrium. On advantage of my approach is that, despite being very general, it is simpler to work with than the univalence or the index theorem approach. The basic application of this theory can be found in chapter three. But it is also worth mentioning that this is the first contribution that separately discusses the scope of multiple symmetric equilibria and the scope of asymmetric equilibria in symmetric games. I show e.g. that my approach gives a new interpretation to the conventional two conditions that assert uniqueness of equilibrium in the Cournot game. Therefore I attribute this chapter an interest in itself.

Finally, this thesis should also make clear that my contribution stands more at the beginning and less at the end of a new and contemporaneously very important line of research. In this sense the final chapter provides a brief overview of further research projects and research questions that are not part of this thesis.

2

Informative Advertising and Limited Attention

In this chapter I investigate the implications of the first stylized fact from the introduction - consumers have limited attention - for the equilibrium of a symmetric oligopolistic model of informative advertising and imperfect economic competition.

I begin by relating informative advertising to the concepts of persuasive and complementary advertising. Then I formulate the general problem of an advertising firm. After discussing advertising technology I integrate limited attention as an upper bound on how many distinct items a consumer may perceive into the model of Grossman and Shapiro (Grossman and Shapiro (1984)). I derive the symmetric equilibrium and discuss the implications of limited attention for equilibrium price, advertising and the degree of diversity.

2.1 Informative advertising

How does advertising affect a firm's demand function? Generally, the literature recognizes three possibilities how advertising interplays with economic competition: advertising may be i) persuasive, ii) informative or iii) complementary in nature (see Bagwell (2007) for a survey). The persuasive view holds that advertising influences consumer's tastes and for example creates brand-loyalty or spurious product-differentiation. Hence advertising tends to make demand more inelastic which leads to higher prices and generally to less competitive markets as advertising may act as an entry-barrier (e.g. Comanor and Wilson (1974)).

The informative view reaches precisely the opposite conclusion. From this perspective advertising does not interact with consumer preferences as the ads convey direct information (e.g. price or location information) to consumers. Advertising hence improves information of consumers which makes demand more elastic and may also facilitate entry as a new firm can broadcast its existence.

The complementary view points out that advertising may be complementary to the advertised products in the sense that although advertising does not change the preference relation the consumption experience of the good may be reinforced by the perception of related advertising (see e.g. Stigler and Becker (1977)). This paper entirely builds on the informative approach to advertising and any reference to advertising refers to informative advertising.

Since Kaldor (Kaldor (1950)), Ozga (Ozga (1960)) and Stigler (Stigler (1961)) the economic importance of advertising was recognised as a mean of information provision which can reduce consumer search costs (Stigler (1961), p.216), increase the adoption rate of new commodities (Kaldor (1950), p. 8) or generally increase information transmission in a society (Ozga (1960), p. 34). Telser (Telser (1964)) was the first to provide empirical evidence that advertising "is an important source of information" (p. 558) and that it promotes competition between advertising firms (p. 550). Nelson (Nelson (1974)), introducing the divison of commodities into search goods and experience goods, also emphasises the informative role of advertising. But in case of experience goods, which require consumption prior to evaluating their quality, advertising also works as a costly signal of product quality. In case of homogeneous goods Butters (Butters (1977)) explores the implications of costly informative advertising on the equilibrium distribution of prices. In the basic model sellers need to advertise in order to sell their goods, as consumers have no information and do not search. Ads are purely informative as they only convey the price of the good to the consumer. In the case where consumers receive ads of more than one firm consumers behave as Bertrand players: they choose the lowest price good and in case of equal prices randomise their choice. As Butters shows this induces firms to play mixed strategies when the number of firms is finite which in the limit case, i.e. if there is an infinity of firms and consumers, leads to a continuous distribution of advertised equilibrium prices. When taken to a model of horizontal product differentiation the complication of mixed strategies does not arise and equilibrium price and advertising are uniquely determined (Grossman and Shapiro (1984)). Their model nicely reveals the pro-competitive potential of informative advertising as firms in markets with greater advertising face more price-elastic demand which leads to lower equilibrium price levels. Price, advertising and entry (market structure) are endogenously determined. In a recent paper, Goeree uses a dataset from the PC industry to show that standard estimation techniques, which assume that the consumers are fully aware of the entire market, systematically and severely overestimate the price-elasticity of demand and underestimate markups and market power of firms (Goeree (2008)). Although Goeree does not incorporate the possibility that consumers have limited attention her estimation results clearly show the importance of understanding how information and economic competition interact with each other.

2.1.1 The general problem of an advertising firm

In this section I state a general version of the problem of an advertising firm that is engaged in imperfect price competition. I assume that consumers receive information by attending to different information channels, e.g. by watching different channels of television, reading different magazines, listening to the radio etc. The countable set of all channels is denoted by $\mathcal{M}$ and $|\mathcal{M}| \leq \infty$. Every channel i has a non-empty audience $\delta(i)$ with size $\delta_i \equiv |\delta(i)|$. There are n active firms indexed by $j = 1, ..., n$ and M_j denotes the advertising campaign of firm j, i.e. $M_j \subset \mathcal{M}$ denotes the subset (media mix) of channels that firm j acquires to broadcast its information. Let $\mathbb{P}(n)$ denote the power set of $\{1, ..., n\}$ and $I_i \in \mathbb{P}(n)$ denotes the information set of consumer i where I_i is the set of all commodities consumer i is informed of. According to the theory of informative advertising firms advertise, i.e. communicate their existence and their price, in order to become a part of I_i. If a firm does not advertise it is not contained in any information set.

Suppose firm j is engaged in imperfect price competition for the budget of those consumers that are informed about j but also of some other firms. The firm must decide how much to advertise and which price to set given the decisions of its competitors. In models of informative price advertising the profit function of firm j takes on the form[1]

$$\Pi^j = (y_j - c_j)\tilde{Q}^j \left(\{M_k, y_k\} : 1 \leq k \leq n\right) - F_j - C_j(M_j) \tag{2.1}$$

where c_j are unit costs of production, F_j is a fixed cost that summarises infrastructure costs for production and advertising, $C_j(M_j)$ is advertising expenditure for campaign M_j, M_{-j} denotes the campaigns of all firms other than j, y_j is the price chosen by firm j and y_{-j} are the prices chosen by all other firms. $\tilde{Q}^j$ is the market demand of firm j. If the firm is not a pure monopolist (see Dorfman and Steiner (1954) for the monopoly case) but an oligopolist $\tilde{Q}^j$ depends on advertising and price choices of all other active firms.

I will use the model of Grossman and Shapiro (Grossman and Shapiro (1984)) to obtain a specific functional form for $\tilde{Q}^j$ and show when and how limited consumer attention affects the demand function. In the model of this chapter each active firm simultaneously[2] and non-cooperatively chooses its price and its advertising campaign. I will also consider the two-stage game where firms decide whether to enter the market or not at the first stage. Those firms who decide to enter then play a simultaneous advertising-pricing game at the second stage. I will discuss how and when limited attention affects the strategic equilibrium of the model. Throughout this chapter I restrict myself to the case of symmetric firms. In the next section I specify advertising technology.

2.1.2 Advertising technology

Suppose firm j chooses campaign M_j. Then the audience of firm j is given by $\bigcup\limits_{i \in M_j} \delta(i)$. I define $\Phi(M_j) \equiv \left| \bigcup\limits_{i \in M_j} \delta(i) \right|$ to be the size of the audience of campaign M_j. Hence $\Phi(M_j)$ depends on the overlap of the audiences $\delta(i)$ for all $i \in M_j$. To illustrate this consider the following two extreme cases. Suppose first that $\delta(i) = \delta(i') = \delta$ for all $i, i' \in M_j$. Hence the audience is the same for all channels of campaign M_j. In such a case we

[1]See e.g. Dorfman and Steiner (1954), Grossman and Shapiro (1984), Esteban et al. (2001), LeBlanc (1998) and also Goeree (2008).

[2]It would be of significant interest to consider a two-stage version of the game where each active firm chooses its advertising campaign at the first stage and its price at the second stage.

have $\Phi(M_j) = |\delta|$. In the opposite case if $\delta(i) \cap \delta(i') = \emptyset$ for all $i, i' \in M_j$, we have $\Phi(M_j) = \sum_{i \in M_j} \delta_i$. Suppose a firm chooses a collection of m channels. Let these channels be indexed by the numbers $1, 2, ..., m$. Then the reach $\Phi(M_j)$ of the information campaign is

$$\Phi(M_j) = |\delta\,(1)| + |\delta\,(2\backslash 1)| + |\delta\,(3\backslash 1 \vee 2)| + ... + |\delta\,(m\backslash(1 \vee 2 \vee ... \vee m-1))| \tag{2.2}$$

where $\delta\,(2\backslash 1) \equiv \delta(2)\backslash\delta(1)$, $\delta\,(3\backslash 1 \vee 2) \equiv \delta(3)\backslash\,(\delta(1) \cup \delta(2))$ and so on. To keep the model tractable I make the following assumption.

Assumption 2.1. *The following three assumptions are imposed:*

i) Every channel i in $\mathcal{M}$ has the same audience size $\delta > 0$. Consumers are randomly distributed among channels.

ii) From the perspective of firm $1 \leq j \leq n$ the conditional probability of a consumer to attend to channel i' given attendance of channel i is the same for all channels:

$$P\,(i'\,|i) = 1 - a_j \quad \forall i \neq i' \in \mathcal{M} \tag{2.3}$$

iii) From the perspective of firm $1 \leq j \leq n$ given that a consumer attends channel $i \in \mathcal{M}$ attending to any two different channels $i', i'' \in \mathcal{M}$ are independent events:

$$P\,(i' \wedge i''\,|i) = P\,(i'\,|i) \cdot P\,(i''\,|i)$$

The next proposition shows that under assumption 2.1 the reach of an information campaign of size m follows a geometric series.

Proposition 2.1. *Under assumption 2.1 the size $\Phi(M_j)$ of the audience of campaign $M_j \in \mathcal{M}$ with $|M_j| = m_j$ is determined by*

$$\Phi(M_j) = \begin{cases} \delta\frac{1-(a_j)^{m_j}}{1-a_j} & a_j < 1 \\ m_j\delta & a_j = 1 \end{cases} \tag{2.4}$$

Proof: Appendix (2.5.1)

Thus we have two parameters, δ and a_j, that are relevant for firm j in determining the size of the audience of its campaign M_j. Whereas δ quantifies the audience of a channel, a_j controls the overlap of the channels and a_j^k can be interpreted as the fraction of consumers of a channel's audience that become aware of the campaign if a campaign of size k is extended by one additional channel. Hence the higher a_j, the less the channels overlap, the more uninformed consumers can be reached by extending the advertising campaign. An interesting interpretation of the parameter a_j is to view it as capturing the ability of firm j to target its messages to previously uninformed consumers. Suppose that for some reason firm j and firm g choose a campaign of size $m_j = m_g = m$ (this does not imply that $M_j = M_g$). This brings to each firm an audience of size $\Phi(M_j)$ and $\Phi(M_g)$. Then $a_j > a_g$ means that firm j can combine the media channels in a more efficient way in the sense that firm j reaches a larger audience than firm g, i.e. $\Phi(M_j) > \Phi(M_g)$, despite of choosing the same campaign size. For simplicity I set $a_j = a \in (0,1)$ for $1 \leq j \leq n$. The introduction of new media, such as the television or the internet, can be thought of as exogenously increasing the parameter δ: more consumers can be reached per channel. At the same it is reasonable to assume that the targeting abilities of the firms also have increased. For example, the use of digital media has made it possible to store certain client characteristics ("cookies") which can be exploited by advertisers.[3]

An interesting case occurs if attending any two channels are independent events. Suppose that every channel reaches a constant fraction r of an overall population of size Δ. Hence $\delta = r\Delta$.

Corollary 2.1. *If $P(i \wedge i') = P(i)P(i')$ and $P(i) = r$ for any $i, i' \in \mathcal{M}$ then* (2.4) *corresponds to the constant-reach-independent-readership (CRIR) information technology.*

Proof:

Because $P(i \wedge i') = P(i)P(i')$ we have $P(i'|\, i) = P(i)$. Because of $P(i) = r$ we have $a = 1 - r$ and with $\delta = r\Delta$ equation (2.4) then reads $\Phi(m) = \Delta\left(1 - (1-r)^m\right)$.

■

[3]A well-known example is Amazon's "People who bought book X also bought ..." page, which a shopper is forced to observe when purchasing book X.

The CRIR technology was first considered by Grossman and Shapiro (Grossman and Shapiro (1984), p. 66). By weakening the assumption of independent readership to independent conditional readership we have gained a further parameter, a, that controls the targeting abilities of firms and is independent of the audience size of a channel. Note that the CRIR technology implies that per-channel reach r and targeting ability a are not independent: a higher reach r automatically implies a lower targeting ability a which seems problematic as the digitalisation of advertising rather suggest that both reach r and targeting abilities a have increased over time. Throughout the paper I assume there is a number of $\Delta > 0$ consumers that are interested in the commodities of firms $1, ..., n$ and $\delta = r\Delta$ with $r \in (0, 1)$. Then we can define the fraction $\phi(m)$ of consumers that are informed about a firm using a campaign of size m by

$$\phi(m) \equiv \begin{cases} r\frac{1-a^m}{1-a} & r\frac{1-a^m}{1-a} \leq 1 \\ 1 & r\frac{1-a^m}{1-a} > 1 \end{cases} \tag{2.5}$$

where $a \in (0, 1)$. From (2.5) we see that $\phi(1) = r$. Hence for $m = 1, 2, ...$ we have $\phi(m) \in [r, K]$ where $K = min\left\{1, \frac{r}{1-a}\right\}$. Let $\phi(m) \in (r, K)$. Then from (2.5) we can deduce that $\phi'(m) > 0$ and $\phi''(m) < 0$.[4] This means that increasing the reach ϕ of an advertising campaign is only possible at a diminishing rate which holds because channels overlap to some degree. It can also be shown (see the appendix) that $\phi_{ma} > 0$ for $m \geq 1$ which means that the marginal change of reach by expanding the campaign is higher if targeting abilities are improved.

Throughout this chapter I assume that all firms have access to the same advertising technology. As Butters (Butters (1977), p. 466) and Grossman and Shapiro (Grossman and Shapiro (1984), p. 66) I assume that channels have a cost proportional to their reach, $\theta r\Delta$, and it is not possible for firm j to anticipate which channels a competitor chooses.[5] Hence $C(M) = C(m) = \theta r\Delta m$ which means that the cost a firm incurs from a campaign M depends only on the size m of the campaign and on the exogenous parameters θ, r and Δ. It will be analytically convenient to formulate the problem of the firm directly in terms of ϕ rather than m. For this purpose I define the function $A(\phi) \equiv \theta r\Delta m(\phi)$ as

[4] $\phi'(m) = -\frac{a^m rLn(a)}{1-a}$ and $\phi''(m) = -\frac{a^m rLn(a)^2}{1-a}$, where $Ln(a) < 0$ since $a < 1$.

[5] This makes sense as choices must be made simultaneously and all channels are identical.

advertising costs of using m channels to reach ϕ consumers.[6] Hence for any firm j the choice of a certain campaign M_j with size m_j uniquely determines the reach $\phi_j = \phi(m_j)$ and also the costs, $A_j = A(\phi_j)$, of the campaign. Note that (2.5) implies that $A'(\phi) > 0$ and $A''(\phi) > 0$ for $\phi \in (0, 1)$.

2.2 Limited attention and informative advertising

Suppose a firm chooses a campaign of size m in order to inform a fraction of $\phi(m)$ consumers of its commodity. Then the firm is present in $\phi\Delta$ information sets and incurs a cost of $A(\phi)$. The conventional model of informative advertising assumes that regardless of the size $|I_i|$ of the information set the consumer evaluates all alternatives he receives information of - even if $|I_i|$ is a large number. As this assumption clearly is inconsistent with the evidence e.g. that consumers only consider the top-positions of search page listings (see chapter 1.2 of this thesis) I henceforth assume that a consumer perceives at most $R_i > 1$ alternatives in making his decision. For simplicity, I set $R_i = R$ and assume that R is commonly known. A consumer with $|I_i| \leq R$ perceives all alternatives in I_i whereas a consumer with $|I_i| > R$ perceives only an R-dimensional subsample $\tilde{A}_i \subset I_i$ of alternatives when making his decision.

As mentioned in the introduction of this thesis Falkinger developed a general equilibrium model of limited attention and economic competition (Falkinger (2008)) with a macroeconomic focus. In his model consumers also have an exogenously fixed threshold on how many different items they can perceive. In both models limited attention may imply that not all alternatives a consumer is informed of are taken into account.[7] One difference of my model lies in the fact that in his model active firms take as exogenously given how many consumers are effectively reached by sending their information. In my model this reach is endogenously determined by strategic advertising decisions of the firms. As in my model the reach decision of the firms is endogenous we can study how limited attention affects the firm's decision to inform a certain amount of consumers.[8]

[6] In case of the specific form (2.5) we have $m = m(\phi, r, a)$ and hence $A(\phi; a) \equiv \theta r \Delta m(\phi, r, a)$.

[7] As we will see this depends on how many senders survive and how strong they advertise and both are determined endogenously by fundamentals.

[8] The other major differences to Falkinger's model are that i) firms are of non-zero mass and engaged in strategic price competition, ii) firms cannot directly influence their chance of perception and iii) my

2.2.1 Limited attention in the Grossman-Shapiro model

I use the generalisation of the Salop model by Grossman and Shapiro (further refered to as GS) to obtain a specific form for $\tilde{Q}^j$ in (2.1) under consideration of limited consumer attention.

Consumers and firms are located uniformly around the circumference of the unit circle. Consumption is a zero-one-decision and consumers have a linear utility function $u_i(j) = v - tw_i(j) - y_j$ where the preference parameter t measures transportation costs per unit of distance and $w_i(j)$ is the shortest arc distance between the location of a firm j with price y_j and the location of consumer i on the circle. Among perceived varieties the consumer always chooses the variety that leaves him with the highest net utility.

$tw_i(j)$ quantifies the disutility a consumer experiences if he chooses to travel a distance of $w_i(j)$ and, following GS, I refer to $tw_i(j)$ as consumer i's transportation costs of consuming at the location of firm j. We can think of t as determining the degree of substitutability between the different varieties (firms) on the circle. In case of a large t the consumer suffers strongly from moving around the circle and has a strong preference of "staying home" (consuming at a location close to his own location). If $t = 0$ then transportation distance is completely irrelevant to the consumer and he will choose to consume at the cheapest of all perceived locations independent of how far he needs to travel. In this border case the varieties are perfect substitutes.

In the game we are about to solve the n active firms simultaneously and non-cooperatively choose their strategy[9], the price-advertising pair (y_j, ϕ_j). I will only consider the case of symmetric firms. Hence formally we have to deal with a static two-dimensional symmetric n-player game. The best way to find a symmetric equilibrium in such a game is to derive the profit function of an individual firm j, which as GS I shall call the representative firm. An interior symmetric equilibrium then can be found by setting $(y, \phi) = (y_1, \phi_1) = ...(y_j, \phi_j) = ... = (y_n, \phi_n)$ in the two first-order conditions of the representative firm's profit function and solving these equations for (y, ϕ).

model is of a partial and not general equilibrium nature. An implication of i) will be, different from Falkinger, that equilibrium prices are a function of the attention threshold R. In chapter 3 I develop a game-theoretic framework of attention competition which will enable me to study how the effort to inform more consumers (to be present in more information sets) and the effort to increase the chance of perception for those consumers that are attention-constrained are interrelated (see chapter 4.2).

[9]I only allow for pure stratgies.

In the following I derive the demand function of firm j, the representative firm. To simplify notation I drop the firm index j whenever there cannot be confusion and let $y = y_j$ and $y_h = \bar{y}$ as well as $\phi_j = \phi$ and $\phi_h = \bar{\phi}$ for any $h \neq j$.[10] The representative firm takes its own location, the locations of the other firms as well as $\bar{y}$ and $\bar{\phi}$ as given.

2.2.1.1 Favourity groups

Fort the rest of section 2.2.1 I will maintain the assumption that $n \geq 2$ and $\bar{\phi} > 0$. In order to derive expected demand for the representative firm I follow the approach of GS and partition the set of consumers into n favourity groups (from the perspective of the representative firm), where group $1 \leq k \leq n$ encompasses all those consumers for which the representative firm produces the k-th favourite variety in terms of net utility (i.e. for fixed locations and given prices y and $\bar{y}$) under full information. For example, a consumer in group $k = 3$ would find the representative firm to be his third-best choice: there are two other firms that offer a higher net utility to this consumer but $n - 3$ firms offer a lower net utility. How many consumers are there in group k? For $k = 1$ all consumers to which the representative firm offers a higher net utility than its closest neighbour[11] will find the representative firm to be their first best choice. Hence the indifferent consumer is located at a distance[12] of w_1 to the representative firm, where w_1 is determined by:

$$v - tw_1 - y = v - t\left(\frac{1}{n} - w_1\right) - \bar{y}$$

which gives $w_1 = \frac{\bar{y}-y}{2t} + \frac{1}{2n}$. Hence all consumers at a distance of $w \leq w_1$ to the representative firm would - if fully informed - choose to consume from the representative firm. Counting consumers on either side of the firm there are $N_1 = 2\Delta w_1$ consumers in this group. Repeating the argument GS (p. 67) then show that the indifferent consumer in

[10] The approach I pursuit here is described generally in chapter 5.2.1.

[11] Remember that we assume all competitors to set the same price $\bar{y}$.

[12] I always assume that $v \geq \max\left\{t, \frac{3}{2}\sqrt{\frac{tF}{\Delta}}\right\}$. As is shown by Salop (Salop (1979), p. 148) a lower bound on v ($v > 3/2\sqrt{tF/\Delta}$) must be imposed in order to obtain the competitive equilibrium configuration under full information. Further, v must be sufficiently large to assure that in the symmetric equilibrium every informed consumer consumes somewhere ($v - y^* - t/2 \geq 0$). As we will see this holds if $v > t$. Finally, all following formulas are valid only in the range $\bar{y} - t/n < y < \bar{y} + t/n$ because then we have $N_k \in (0, 1)$ for all k.

group $k = 2, ..., n-1$ is at a distance w_k determined by

$$v - tw_k - y = v - t\left(\frac{k}{n} - w_k\right) - \bar{y}$$

Hence there are

$$N_k = 2\Delta\left(w_k - w_{k-1}\right) = 2\Delta\frac{1}{2n} = \frac{\Delta}{n} \qquad k = 2, ..., n-1$$

consumers in these groups. Finally, N_n is the number of consumers not in the other $(n-1)$ groups:

$$N_n = \Delta - \sum_{k=1}^{n-1} N_k = \Delta - (n-2)\frac{\Delta}{n} - N_1 = \frac{\Delta}{n} - \frac{\Delta(\bar{y} - y)}{t}$$

2.2.1.2 Expected demand under limited attention

I now derive expected market demand of the representative firm under limited attention. Let $E\left[q_k, R\,|j \in I_k\right]$ denote the representative firm's expected demand of a member of group k conditional on $j \in I_k$.[13] Because $E\left[q_k, R\,|j \notin I_k\right] = 0$ and $Prob(j \in I_k) = \phi$ the law of total expectations implies $E\left[q_k, R, \phi\right] = \phi E\left[q_k, R\,|\phi\right]$ where $E\left[q_k, R, \phi\right]$ denotes (unconditional) expected demand from a member of group k. Hence expected market demand is

$$\begin{aligned} E[Q, R] &= \sum_{k=1}^{n} \phi E\left[q_k, R\,|j \in I_k\right] N_k \\ &= \phi\Delta\left(\tfrac{\bar{y}-y}{t} + \tfrac{1}{n}\right) E\left[q_1, R\,|j \in I_1\right] + \tfrac{\phi\Delta}{n} \sum_{k=2}^{n-1} E\left[q_k, R\,|j \in I_k\right] \\ &\quad + \phi\Delta\left(\tfrac{1}{n} - \tfrac{\bar{y}-y}{t}\right) E\left[q_n, R\,|j \in I_n\right] \end{aligned} \tag{2.6}$$

Note that expected market demand of the representative firm as defined by (2.6) is actually a function of $(\phi, y, \bar{y}, \Delta, t, n, R)$ but I will refer to this demand function by $E[Q, R]$ to keep notation simple.

The remainder of this section is concerned with deriving an expression for $E\left[q_k, R\,|j \in I_k\right]$. The formal derivation can be found in the appendix (see 2.5.3). In the main text I limit

[13] The notation $j \in I_k$ means, in slight abuse of how I defined an information set, that the representative firm has reached a member of group k, i.e. is part of this consumers' information set. This notation is succinct for $j \in I_i$ where $i \in$ group k.

myself to providing the basic intuition.

The fraction of consumers who are informed of all competitors of the representative firm is $\bar{\phi}^{n-1} > 0$. Now suppose that $n > R$. Then there must be some consumers who receive information of at least R competitors. If such a consumer receives an ad also from the representative firm then this consumer's attention constraint is binding and he is not able to perceive all information. But there are also consumers who receive only little information. For example, the fraction of consumers not informed about any competitor of the representative firm is $(1-\bar{\phi})^{n-1} > 0$. Consumers informed about the representative firm but about less than R competitors are capable of perceiving all information they received. The representative firm cannot deduce whether a specific consumer's attention constraint is binding or not. But for given $\bar{\phi}$, n and R the firm can form an expectation whether a consumer has limited attention (a binding attention constraint) or limited information (a non-binding constraint). This distinction is important because the firm's expected demand from a consumer depends on the probability of this consumer to be attention constrained or not.

In the appendix (see 2.5.3) I show that if for those consumer with limited attention the subset $\tilde{A}_k$ of perceived alternatives is randomly selected[14] from I_k then $E\left[q_k, R \,|\, j \in I_k\right]$ is determined by

$$
\begin{aligned}
E\left[q_k, R \,|\, j \in I_k\right] &= \underbrace{\sum_{z=0}^{R-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix}}_{B_1} \\
&+ \underbrace{\sum_{z=R}^{n-1} \sum_{s=0}^{k-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} k-1 \\ s \end{pmatrix} \begin{pmatrix} n-k \\ z-s \end{pmatrix} \frac{\begin{pmatrix} z-s \\ R-1 \end{pmatrix}}{\begin{pmatrix} 1+z \\ R \end{pmatrix}}}_{B_2}
\end{aligned} \tag{2.7}
$$

where I set $B_2 = 0$ if $R \geq n$. In this expression B_1 corresponds to the firm's conditional demand if the consumer has limited information (if he receives ads of less than R opponents). B_2 corresponds to the firm's conditional demand if the consumer has limited

[14] In chapter 3 I develop the machinery required to incorporate the possibility that, given a consumer has been reached by a firm, the firm can influence its chance of perception if the consumer has received information of more than R firms. Equipped with these tools I shall return to this model in chapter 4.2.

attention (if he receives ads of at least R opponents). Note that $E\left[q_k, R\,|j \in I_k\right]$ does not depend on y nor on ϕ but it depends on the reach of the competitors $\bar{\phi}$ and also on R, k and n. We will see later that limited attention plays an important role for the strategic choice of prices and advertising if the consumers are attention constrained on average. For expression (2.7) this intuitively means that B_2 is the predomiant term.

2.2.2 Equilibrium conditions

In this section I derive and discuss the two equilibrium conditions of the symmetric price-advertising n-firm game. To gain as much intuition as possible I will discuss separately i) how limited attention affects the equilibrium price if we take advertising of all firms as exogenously given and ii) how the optimal choice of advertising for given prices of the representative firm depends on $\bar{\phi}$. We will see that limited attention ceteris paribus implies that a higher equilibrium price can be sustained for any exogenously given amount of advertising and that the representative firm's optimal choice of advertising, if all prices are exogenously given and identical, does not depend on limited attention.

With $E\left[q_k, R\,|j \in I_k\right]$ as defined by (2.7) and $\tilde{Q}^j = E\left[Q, R\right]$ as defined by (2.6), $F_h = F > 0$ and $c_h = c \geq 0$ for all $h = 1, ..., n$ and the definition of $A(\phi)$ the general profit function in (2.1) simplifies to[15]

$$\begin{aligned} \Pi(y,\phi) &= (y-c)E\left[Q,R\right] - F - A(\phi) \\ &= (y-c)\phi\Delta\left[\left(\tfrac{\bar{y}-y}{t} + \tfrac{1}{n}\right)E\left[q_1, R\,|j \in I_k\right] + \tfrac{1}{n}\sum_{k=2}^{n-1} E\left[q_k, R\,|j \in I_k\right]\right. \\ &\quad \left. + \left(\tfrac{1}{n} - \tfrac{\bar{y}-y}{t}\right)E\left[q_n, R\,|j \in I_k\right]\right] - F - A(\phi) \end{aligned} \tag{2.8}$$

The representative firm takes as given $\left(\bar{y}, \bar{\phi}, c, t, F, a, r, \theta, \Delta, n, R\right)$ and simultaneously chooses[16] (y, ϕ) in order to maximise (2.8). From (2.8) and (2.35) we see that the price $\bar{y}$ affects $E[Q, R]$ and the profit in a linear way and that campaign choices of the opponents matter only up to their size.[17]

To derive the equilibrium conditions I require the following lemma. If $R' > R \geq n$ we can see from (2.35) and (2.6) that $E\left[Q, R\right] = E\left[Q, R'\right] = E\left[Q, \infty\right]$. To keep notation

[15] This gives the profit function in what I call the symmetric opponent form. See chapter 5.

[16] Note that (2.8) is twice differentiable in y and ϕ. Second-order conditions of this optimisation problem are satisfied. See section 2.5.7 in the appendix.

[17] Because $\bar{\phi} = \phi(\bar{m})$.

simple I set $E[Q] \equiv E[Q, \infty]$ as the expected demand for given $(y, \phi, \bar{y}, \bar{\phi})$ that results in the absence of limited attention.

Lemma 2.1. *Let $R, R' > 1$ and $\bar{y} = y$. Then $E[Q, R'] = E[Q, R]$ and also $\frac{\partial E[Q,R]}{\partial \phi} = \frac{\partial E[Q,R']}{\partial \phi}$ with $E[Q, R] = E[Q] = \frac{\phi \Delta}{n \bar{\phi}} \left(1 - (1 - \bar{\phi})^n\right)$.*

Proof: Appendix (2.5.5)

Lemma 2.1 simply says that for equal prices and given $\phi, \bar{\phi}$ expected demand as well as expected marginal demand (with respect to ϕ) of the representative firm are independent of R. Intuitively, this must hold because $\tilde{A}_k \subset I_k$ consists of R randomly drawn firms for those consumer with $|I_k| > R$. In case of identical prices the expected demand of the firm cannot depend on R. If this were the case then some consumers would systematically have to ignore the information of a certain firm which contradicts the assumption of randomisation. Assuming that $(1-\bar{\phi})^n \approx 0$ we see from lemma 2.1 that the representative firm's demand as well as marginal demand depend negatively on $\bar{\phi}$. If the competitors of the representative firm advertise more ceteris paribus then the consumers that are aware of the representative firm are aware of more competitors on expectation. Hence in the groups with $k > 1$ the chance that the consumer perceives a superior firm is increased which means that for given prices the representative firm realises fewer sales. As a general comment it should be noted that there are important reasons why we could expect informative advertising to impose a *positive* externality on firms. As Telser (Telser (1964), p. 540) argues in the absence of significant product differentiation there will not be much advertising as advertising does not shift own demand by much but nevertheless e.g. a local bakery might profit from an overregional promotion of bread. The high availability of information on the internet and search pages such as Google might also be a reason why *offline* advertising might impose a positive externality. If someone who hears about a new commodity over conventional channels enters a query in a search filter as Google then this person may be confronted immediately with an abundance of related commodities and end up by choosing a different brand or buys an additional related product he found among the links.[18]

[18]The implications of this type of externality in the context of limited attention is subject to my current research mimeo "Attention markets".

2.2.2.1 The equilibrium price under exogenous advertising

Suppose all active firms simultaneously choose their price but take advertising as exogenoulsy given: $\phi = \bar{\phi} > 0$. In a symmetric equilibrium of this one-dimensional pricing game the equilibrium price y can be found differentiating (2.8) with respect to y and solving this equation for y. Using lemma 2.1 this gives

$$\left.\frac{\partial \Pi(y,\phi)}{\partial y}\right|_{y=\bar{y}} = \frac{\Delta}{n}\left(1-(1-\bar{\phi})^n\right) + \frac{\bar{\phi}\Delta(y-c)}{t}\left(-E[q_1, R\,|j \in I_1] + E[q_n, R\,|j \in I_n]\right) = 0 \tag{2.9}$$

I show in the appendix (see 2.5.6) that

$$y = c + \frac{t}{n\bar{\phi}} \frac{\left(1-(1-\bar{\phi})^n\right)}{\left(1-(1-\bar{\phi})^{n-1}-\lambda\right)} \tag{2.10}$$

where

$$\lambda \equiv \begin{cases} \sum\limits_{z=R}^{n-1}\left(\bar{\phi}^z(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\left(1-\frac{R}{1+z}\right)\right) > 0 & \text{if } R < n \\ 0 & \text{else} \end{cases} \tag{2.11}$$

with

$$1-(1-\bar{\phi})^{n-1}-\lambda > 0 \tag{2.12}$$

If $R \geq n$ (2.10) corresponds to the (unapproximated) price equation from GS (p.69). From (2.11) we see that with limited attention we have $\lambda \in (0,1)$. A higher value of λ reduces the denominator which means that the representative firm ceteris paribus chooses to set a higher price under limited attention (if $n > R$) compared to a situation without limited attention ($R \geq n$). Also note that

$$R' < R \leq n \Rightarrow y(R') > y(R) \tag{2.13}$$

This can be seen from (2.11) as a higher value of R means that the sum gets shorter and the factor $\left(1-\frac{R}{1+z}\right)$ gets smaller. Hence λ decreases and the denominator in (2.10) increases. Why is the price higher under limited attention ceteris paribus? Under limited attention consumers who are informed of the representative firm but also of at least R

other firms do not compare all alternatives. This implies that demand must become less price elastic: if the representative firm sets a (marginally) higher price it does not loose as many consumers as under unlimited attention because some consumers might ignore the superior firms (after the price increase). This intuitively explains why a higher equilibrium price can be sustained for any given $\bar{\phi}$ under limited attention. Figure 2.1 depicts the price as determined by (2.10) as a function of $\bar{\phi}$ for given t, n and c.[19] In

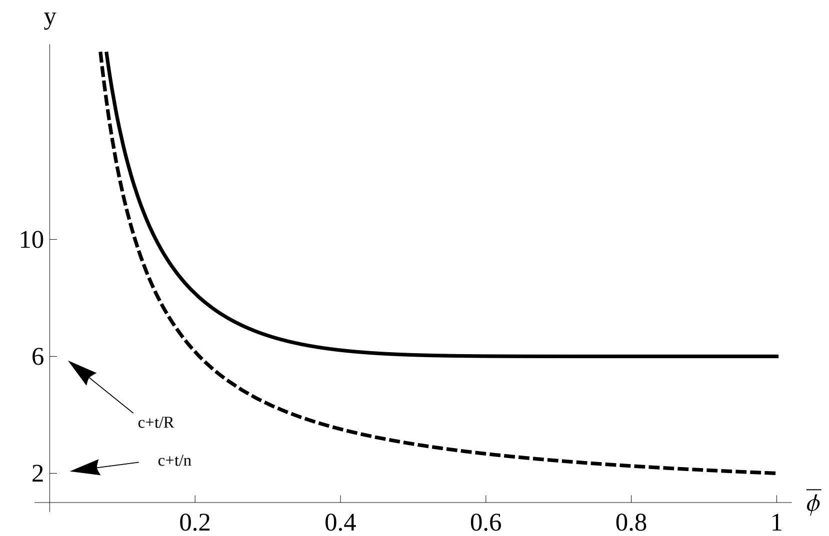

Figure 2.1: Pricing locus with limited (solid line) and unlimited attention

figure 2.1 the solid line represents the case where $n > R$ and the dashed line represents the case where $R \geq n$. We see that in both cases y decreases in $\bar{\phi}$. This is the information effect. To understand this effect intuitively note that a higher $\bar{\phi}$ means that consumers who are informed about the representative firm are informed of more competitors. This decreases the chance of the representative firm to be the perceived superior firm. Hence the representative firm, for given y, looses some consumers. By lowering its price y the representative firm can counteract this loss of demand as this increases the number of consumers who find the representative firm to be their first-best choice. This explains intuitively why the equilibrium price decrease in $\bar{\phi}$.

In figure 2.1 we see that the gap between the solid and the dashed line increases as $\bar{\phi}$ increases which also makes sense intuitively. If $\bar{\phi}$ is small the fraction of people who are informed about the representative firm and also about at least R competitors is

[19]I set $t = 10$, $n = 10$, $c = 1$ and $R = 2$.

small as the firms do not advertise a lot. Many consumers have $|I| \leq R$ which means that the representative firm is especially concerned with information-poor consumers. In expression (2.7) this corresponds to the case where B_1 is predominant and explains why the solid and the dashed line are close together at small levels of $\bar{\phi}$. If $\bar{\phi}$ is large then many consumers have $|I| > R$ so that the demand from these consumers are the main concern of the representative firm. Suppose we have $\bar{\phi} \approx 1$. Then (see the appendix for the derivation)

$$y\left(\bar{\phi} \approx 1\right) = \begin{cases} c + \frac{t}{R} & R < n \\ c + \frac{t}{n} & R \geq n \end{cases} \tag{2.14}$$

If $\bar{\phi} \approx 1$ almost all consumers that are informed about the representative firm are also informed about its $n-1$ competitors. Suppose that $R \geq n$. We see from (2.7) that

$$E\left[q_k, R\,|j \in I_k\right] = \sum_{z=0}^{n-1} 1^z 0^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix} = \begin{pmatrix} n-k \\ n-1 \end{pmatrix}$$

Hence $E\left[q_k, R\,|j \in I_k\right] = 1$ if and only if $k = 1$ and $E\left[q_k, R\,|j \in I_k\right] = 0$ otherwise. Intuitively, this occurs because $\bar{\phi} \approx 1$ and $R \geq n$ imply that, for given prices y and $\bar{y}$, the representative firm can only make a sale to his prime segment (the group $k = 1$) as any consumer belonging to a different group must perceive a superior offer almost surely. Then (2.6) becomes $E\left[Q, R\right] = \Delta\left(\frac{\bar{y}-y}{t} + \frac{1}{n}\right)$ which corresponds to the conventional demand function of the Salop model (Salop (1979), p. 144). We see from figure 2.1 (or from (2.10)) that y converges towards $c + t/n$ if $\bar{\phi} \to 1$. However, if $n > R$ we have $y = c + t/R > c + t/n$. Intuitively, this must hold because what determines the markup if $\bar{\phi} \approx 1$ is the *perceived* number of firms. If $n > R$ and $\bar{\phi} \approx 1$ then most consumers perceive R firms which means that the representative firm effectively has to compete only with $R-1$ competitors for the consumers who are informed about the representative firm. It is as if the market consisted only of R firms which are located around the circle with an average distance of $1/R$ between firms.

To approximate their solution GS assume that the expressions $(1-\bar{\phi})^n$ and $(1-\bar{\phi})^{n-1}$ are negligible. If this is applied to (2.11) we get for $n > R$ (see the appendix)

$$1 - \lambda \cong \frac{R}{n\bar{\phi}} \tag{2.15}$$

Using the two GS approximations and (2.15) in (2.10) gives

$$y \cong \begin{cases} c + \frac{t}{R} & R < n \\ c + \frac{t}{n\bar{\phi}} & R \geq n \end{cases} \tag{2.16}$$

If we use the approximation (2.16) we see that we possibly can have $y(R < n) < y(R' > n)$. This contradicts the general finding of (2.13) and originates in the usage of the approximation. Hence it makes sense to restate (2.16) as

$$y \cong \begin{cases} c + \frac{t}{R} & R < n\bar{\phi} \\ c + \frac{t}{n\bar{\phi}} & R \geq n\bar{\phi} \end{cases} \tag{2.17}$$

The approximation requires $(1-\bar{\phi})^n$ to be small. But if $(1-\bar{\phi})^n$ becomes smaller then $n\bar{\phi}$ becomes larger. The expression $n\bar{\phi}$ approximately measures of how many varieties a consumer is informed of on average. Hence the requirement that $R < n\bar{\phi}$, which if satisfied always implies $R < n$, means that the attention constraint is binding on average. Then scarce consumer attention and not scarce information is the regular case in the economy and it intuitively makes sense to work with (2.17) instead of (2.16) and I will do so in the remainder of this chapter.

2.2.2.2 Optimal advertising for given prices

Now suppose that prices are exogenously given and $y = \bar{y}$. The representative firm can only choose ϕ and takes prices as well as $\bar{\phi}$ as given. The first-order condition of the representative firm then is

$$\frac{\partial \Pi(y,\phi)}{\partial \phi} = (y-c)\frac{E[Q,R]}{\phi} - A'(\phi) = 0 \tag{2.18}$$

Lemma 2.1 implies

$$(y-c)\frac{\Delta}{n\bar{\phi}}\left(1-(1-\bar{\phi})^n\right) = A'(\phi)$$

or with $(1-\bar{\phi})^n \approx 0$

$$(y-c)\frac{\Delta}{n\bar{\phi}} = A'(\phi) \tag{2.18'}$$

Note from (2.18') that limited attention plays no role for the optimal choice of ϕ as R does not occur in this equation for exogenously given prices. The intuition of this result is the same as the intuition behind lemma 2.1: if attention constrained consumers overlook information at random then, for equal prices, more advertising by the representative leads to the same increase of the representative firm's expected demand for any choice of $R > 1$.[20]

2.2.2.3 Limited attention and the Dorfman-Steiner theorem

From (2.8) we can deduce a simple relation between advertising and price. Let

$$\varepsilon_y \equiv \left| \frac{\partial E[Q,R]}{\partial y} \frac{y}{E[Q,R]} \right|$$

denote price elasticity of demand and $\varepsilon_A \equiv \frac{\phi A'(\phi)}{A(\phi)}$ is the elasticity of reach costs $A(\phi)$. Then using the two first-order conditions of (2.8) it is an easy exercise to show that

$$\frac{1}{\varepsilon_y \varepsilon_A} = \frac{A(\phi)}{yE[Q,R]} \tag{2.19}$$

which can be considered a variant of the Dorfman-Steiner theorem (Dorfman and Steiner (1954), p. 828) which relates price and advertising elasticity of demand to each other. If the ratio of advertising expenditure $A(\phi)$ to (expected) sales revenue $yE[Q,R]$ is interpreted as advertising intensity, then we see from (2.19) that if price elasticity of demand decreases, ceteris paribus, then the optimal advertising intensity is increased.

If we evaluate (2.19) in case of our model for $y = \bar{y}$ we approximately end up with (see 2.5.10 in the appendix for details)

$$\frac{A(\phi)}{\bar{y}E[Q,R]} = \begin{cases} \frac{t}{\varepsilon_A \bar{y} R} & R < n\bar{\phi} \\ \frac{t}{\varepsilon_A \bar{y} n\bar{\phi}} & R \geq n\bar{\phi} \end{cases} \tag{2.20}$$

which shows that limited attention ceteris paribus implies a higher advertising intensity because demand is less price elastic. From this expression we see that the GS conjecture after which more average information (measured by a higher $n\bar{\phi}$) raises elasticity of

[20] As we will see in chapter 4.2 this statement may change if firms can effectively *compete* for consumer attention.

demand (p. 69) does not hold under limited attention.

2.3 The symmetric equilibrium

In this section I derive and discuss the symmetric equilibrium of the symmetric n-player game if firms simultaneously and non-cooperatively choose price y and reach ϕ.

2.3.1 Equilibrium for given diversity

Suppose the number of active firms, n, is exogenously given. In the circular model n is a measure of diversity in the economy (Salop (1979)). More active firms means a higher degree of diversity.[21]

The model then has the following set of parameters: $\mathcal{F} = \{F, t, n, R, \Delta, c, \theta, r, a\}$. The vector (y, ϕ) is endogenously determined in a symmetric Nash-equilibrium. As mentioned in section 2.2.1 the equilibrium conditions that determine (y, ϕ) in a symmetric equilibrium can be derived from the first-order conditions of the representative firm's optimization problem. Under the assumption that a symmetric equilibrium with $\phi \in (r, K)$ with $K = min\left\{1, \frac{r}{1-a}\right\}$ exists we can evaluate (2.17) and (2.18') at $\bar{\phi} = \phi$ and end up with the following two approximate equilibrium conditions:

$$y = \begin{cases} c + \frac{t}{R} & R < n\phi \\ c + \frac{t}{n\phi} & R \geq n\phi \end{cases} \tag{2.21}$$

$$(y - c)\frac{\Delta}{n\phi} = A'(\phi) \tag{2.22}$$

The solution (y, ϕ) to (2.21)-(2.22) is a good approximation to the true equilibrium whenever $(1 - \phi)^n \cong 0$ (or equivalently $n\phi$ is large). For simplicity I refer to (2.21)-(2.22) as equilibrium conditions.

[21] If we randomly draw a consumer on the circle then the expected distance to the closest firm is $1/2n$ as consumers are located uniformly around the circle. If n increases this distance is reduced. This means that, on average, a higher n implies a better match between consumers and firms: the average distance between the "ideal" variety of a consumer (which corresponds to the location of the consumer on the circle) and the closest variety decreases with n.

Define a function $\psi(\phi)$ by

$$\psi(\phi) \equiv \begin{cases} \frac{t\Delta}{Rn\phi} - A'(\phi) & R < n\phi \\ \frac{t\Delta}{(n\phi)^2} - A'(\phi) & R \geq n\phi \end{cases} \tag{2.23}$$

Proposition 2.2. *Suppose $\psi(r) > 0$ and $\psi(K) < 0$ where $K = min\left\{1, \frac{r}{1-a}\right\}$. Then there exists a unique solution to (2.21)-(2.22) and $\phi \in (r, K)$ is determined by*

$$A'(\phi) = \begin{cases} \frac{t\Delta}{Rn\phi} & R < n\phi \\ \frac{t\Delta}{(n\phi)^2} & R \geq n\phi \end{cases} \tag{2.24}$$

Proof: Appendix (2.5.11)

Suppose a solution $\phi \in (r, K)$ to (2.24) exists. Then we either have $n\phi \leq R$ or $n\phi > R$. I call the first type of equilibrium an information equilibrium[22] and the second type an attention equilibrium.[23] Which type occurs is endogenously determined and depends on the set of exogenous parameters in $\mathcal{F}$. Suppose an economy has a set of parameters such that an attention equilibrium occurs endogenously. I call such an economy an attention economy. I call an economy with parameters such that an information equilibrium occurs information economy. The following lemma shows how the parameters in $\mathcal{F}$ influence whether an economy is an information or an attention economy. Consider the following expedient system of equations

$$\begin{aligned} &A'\left(\phi^N\right) = \frac{t\Delta}{(n\phi^N)^2} \\ &\Omega^N = \phi^N n \end{aligned} \tag{2.25}$$

Note that the first equation of (2.25) corresponds to (2.24) if $R \geq n\phi$.

Lemma 2.2 (Regime type)**.** *Suppose parameters in $\mathcal{F}$ are such that a unique solution $\phi \in (r, K)$ to (2.24) exists. If (2.25) has a solution $\left(\phi^N, \Omega^N\right)$ with $\Omega^N \leq R$ then an information equilibrium occurs and $\phi = \phi^N$. If $\Omega^N > R$ then an attention equilibrium occurs and $\phi > \phi^N$. A necessary condition for an attention equilibrium is that $n > R$. For fixed R we have*

$$\Omega^N = \Omega\left(\underset{+}{t}, \underset{-}{\theta}, \underset{+}{r}, \underset{+}{a}, \underset{+}{n}, R\right) \tag{2.26}$$

[22]The information equilibrium corresponds to the conventional equilibrium of this model.

[23]Falkinger calls the first type of equilibrium "information-poor" and the second type "information-rich" (Falkinger (2008)).

Proof: Appendix (2.5.12)
I will discuss the causes of an attention equilibrium in more detail in the next section where n is determined endogenously.

From (2.21) we see that in the model equilibrium prices depend on i) the preference parameter t, ii) the technology parameter c and iii) the attention limit R. If $t > 0$ we have $y > c$. This is the case because $t > 0$ means that the varieties are not perfect substitutes which implies that the firms are engaged in an imperfect economic competition. The less the varieties are substitutable (the higher t is) the less severe is the economic competition and the higher are equilibrium markups. This is a standard result of imperfect competition. What is non-standard is the dependence of the equilibrium price on R, a psychological parameter.[24] If parameters are such that an attention equilibrium occurs ($n\phi > R$), then equilibrium prices are determined independently of $n\phi$. Any change of parameters other than c, R or t has no effect on equilibrium prices as long as the change of parameters does not change the attention economy into an information economy. The reason is that if consumer attention is scarce on average, the average perceived market size is R and not $n\phi$. But from the perspective of the competing firms only the market as perceived by consumers is important. This indicates a fundamental difference between an attention economy and an information economy. I will return to this point in the next section.

To see the consequences of limited attention on equilibrium reach ϕ I investigate the comparative statics of R. In figure 2.2 I depict ϕ and y as functions of R holding all other parameters fixed. In this figure I assume parameter values such that for $R < \tilde{R}$ an attention equilibrium occurs. From the figure we see that limited attention, ceteris paribus, implies that firms choose higher prices but also to inform more consumers[25]. But R does not enter equation (2.22) directly. Hence the fact that $\phi'(R) < 0$ if $n\phi > R$ is explained only by the indirect effect of R on ϕ over y: because $y'(R) < 0$ the equilibrium markup is higher under limited attention which implies a higher marginal revenue on advertising (as on average more funds can be extracted from every additionally informed

[24]This is a difference to the model of Falkinger where equilibrium prices depend only on the demand elasticity and unit costs irrespective of the scarcity regime (Falkinger (2008)).

[25]From (2.24) we get $\phi'(R) = -\frac{A'/R}{A'/\phi + A''} < 0$ for $R < \tilde{R}$.

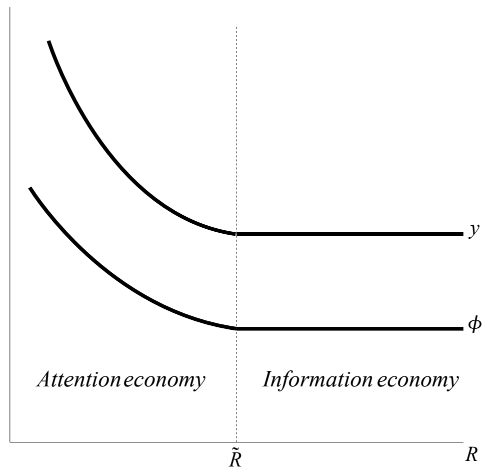

Figure 2.2: The impact of limited attention on ϕ and y

consumer) which increases equilibrium advertising effort. In his paper Telser conjectures that advertising is high if the firm is large with respect to the market as only in this case profits can be increased by advertising more than expenditures (Telser (1964), p.540). My contribution suggests that this statement must be revised under limited attention: if consumers only consider a part of all their information this makes the relevant market for a firm much smaller and the incentive to advertise may still be very high even if n is large.

2.3.2 Endogenous diversity under free entry

Suppose now that equilibrium diversity n is endogenously determined by the zero-profit condition. The set of exogenous parameters of the model then is $\mathcal{B} = \{F, t, R, \Delta, c, \theta, r, a\}$. The approximate zero-profit condition (evaluate (2.8) at $\bar{y} = y$ and $\bar{\phi} = \phi$ and set $(1 - \phi)^n = 0$) is

$$\Pi = (y - c)\frac{\Delta}{n} - F - A(\phi) = 0 \tag{2.27}$$

But using (2.27) with (2.22) gives

$$\Pi(\phi) = \phi A'(\phi) - F - A(\phi) = 0 \tag{2.27'}$$

The three equations[26] (2.21), (2.24) and (2.27') determine (y, ϕ, n) as a function of the exogenous parameters in $\mathcal{B}$. I refer to these equations as the equilibrium conditions of the free entry game. The equilibrium ϕ is determined only by (2.27') and hence by the parameters $\{F, r, \theta, a, \Delta\}$. Note that neither R nor n appear in (2.27'). If $n\phi \leq R$, average information per consumer is scarce, the conventional equilibrium of GS occurs. As in the last section I refer to an equilibrium with $n\phi \leq R$ as an information equilibrium. If however $n\phi > R$, so that average attention per consumer is scarce, then an attention equilibrium occurs. If ϕ as determined by (2.27') is plugged into (2.24) the equilibrium degree of diversity n is determined. Let $\Omega \equiv n\phi$. Then by (2.24) and (2.27') we have $\Omega\,(t, \Delta, F, \theta, r, a, R)$. Consider the following expedient system:

$$\begin{array}{l} \phi^N A'(\phi^N) = F + A(\phi^N) \\ \frac{t\Delta}{(n^N\phi^N)^2} = A'(\phi^N) \\ \Omega^N = \phi^N n^N \end{array} \tag{2.28}$$

Lemma 2.3 (Regime type under free entry)**.** *Suppose* (2.24) *and* (2.27') *have a unique solution* (ϕ, n) *with* $\phi \in (r, K)$ *where* $K = min\left\{1, \frac{r}{1-a}\right\}$. *If the system* (2.28) *has a solution* (ϕ^N, n^N, Ω^N) *with* $\Omega^N > R$ *then we have* $\phi n > R$*, i.e. an attention equilibrium occurs. If* $\Omega^N \leq R$ *then we have* $\phi n = \phi^N n^N \leq R$*, i.e. an information equilibrium occurs. Moreover, we have*

$$\Omega^N = \Omega\left(\underset{+}{t}, \underset{+}{\Delta}, \underset{-}{F}, \underset{-}{\theta}, \underset{+}{r}, \underset{+}{a}, R\right) \tag{2.29}$$

<u>Proof</u>: Appendix (2.5.13)

Lemma 2.3 means that an attention equilibrium occurs whenever a conventional equilibrium would imply more average information $n\phi$ per consumer than the consumer can perceive. From (2.29) we see the causes of an attention equilibrium (for fixed R). From a technological perspective more efficient or cheaper advertising technology (as caused by lower reach costs θ, a higher reach per channel r or better targeting ability a), a larger potential audience Δ and lower overall setup costs F increase the likelihood that an attention equilibrium occurs. A modern economy appears to satisfy these requirements.

[26]Formally, we now have a two-stage game where at the first stage firms decide to enter and pay the setup costs F and then play a game according to section 2.3.1 given that n firms have decided to enter.

For example, the introduction of modern information technologies, as satellite television or the internet (see figure 1.5 of the introduction), certainly implies that a larger (e.g. international) potential audience can be addressed or that the reach r per media channel is increased. It is also argued that the Internet may substantially decrease both the cost of information provision and infrastructure costs: Wal-mart built 276 stores before it reached one billion dollars in sales whereas Amazon needed six warehouses to service over three billions in North American sales in 2003 (Ellison and Ellison (2005), p. 149). Concerning economic competition the model suggests that less intense economic competition (higher market power of the firms), because varieties are less substitutable (a higher t), is more likely to induce an attention equilibrium. Generally, the causes of an attention equilibrium are very similar to the causes of an information-rich economy as characterised by Falkinger (Falkinger (2008), p. 1606).

I now discuss the implications of limited attention on the equilibrium vector (y, ϕ, n).

Proposition 2.3. *If* $\Pi(r) < 0$ *and* $\Pi(K) > 0$ *(see* (2.27')*), where* $K = min\left\{1, \frac{r}{1-a}\right\}$, *then there exists a unique solution* (y, ϕ, n) *to* (2.21), (2.24) *and* (2.27') *with* $\phi \in (r, K)$. *If an attention equilibrium occurs (*$\Omega^N > R$*) then* $y'(R) < 0$ *and* $n'(R) < 0$.

Proof: Appendix (2.5.14)

To discuss the consequences of limited attention for the strategic equilibrium I compare two almost identical economies.[27] Let $\mathcal{E}_1 = \{F, t, \Delta, c, \theta, r, a, R_1\}$ and $\mathcal{E}_2 = \{F, t, \Delta, c, \theta, r, a, R_2\}$. The only difference is that $R_1 < R_2 = \infty$. Hence $\mathcal{E}_2$ corresponds to the conventional information economy of GS which *by assumption* excludes limited attention. From proposition 2.3 and lemma 2.3 we know that $\phi_1 = \phi_2 = \phi^N$, provided a solution to (2.27') exists. Suppose that a solution exists and $\phi^N \in (r, K)$. By lemma 2.3 we then have $n_2 = n^N$. Assume $R_1 < \Omega^N < R_2$ so that an attention equilibrium occurs endogenously in economy one. I will refer to economy one as the attention economy and call economy two information economy. Define

$$\kappa \equiv \begin{cases} \frac{t\Delta}{nR\phi} n & R < n\phi \\ \frac{t\Delta}{(n\phi)^2} & R \geq n\phi \end{cases}$$

Figure 2.3 illustrates how equilibrium diversity n_1 and n_2 are determined in the two economies. As $n^N > R_1/\phi^N$ an attention equilibrium occurs and n_1 corresponds to

[27] In fact, the following is a comparative-static exercise over R.

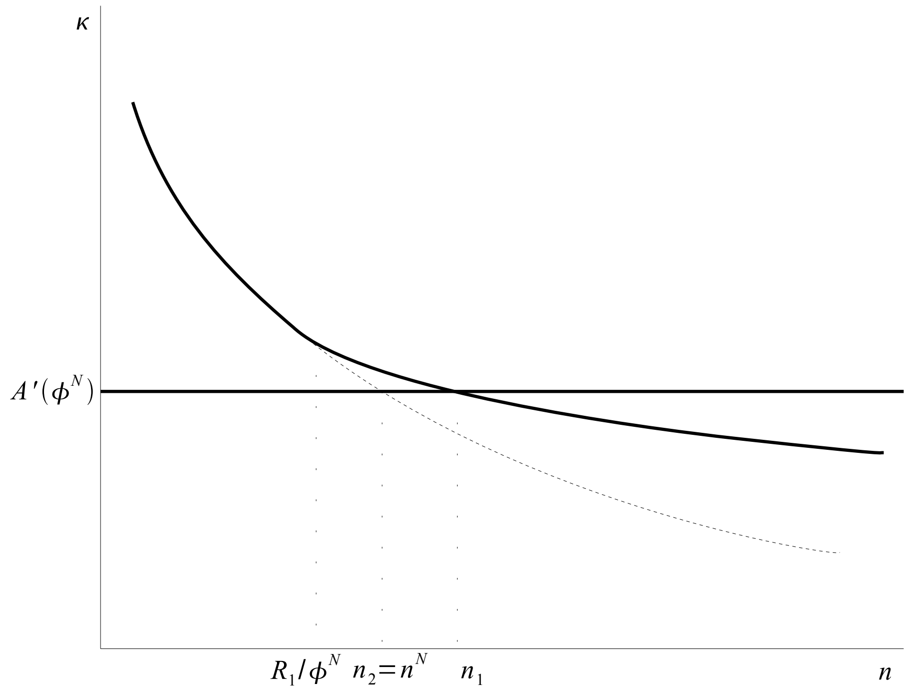

Figure 2.3: Equilibrium diversity in the attention and the information economy

the equilibrium level of diversity of the attention economy. Because $n_1 > n^N$ we see that limited attention, ceteris paribus, leads to a higher degree of equilibrium diversity compared to the solution of the information economy. The number of informed consumers per firm is the same in both economies ($\phi_1 = \phi_2$) but the exposure of a consumer to information is larger in economy one ($n_1\phi_1 > n_2\phi_2$). From (2.21) we then see that $y_1 > y_2$. Why do we have more diversity, higher per consumer information exposure and higher prices in the attention economy? In an attention equilibrium attention is scarce which means that perceived and not effective diversity matters for consumer choice. The average consumer i compares only the R alternatives in $\tilde{A}_i$ and not the $n\phi$ alternatives in I_i to each other. For this reason the strategically behaving firms can set higher markups which translate into higher revenues. But increased revenues promote entry which means more active firms and hence a higher degree of equilibrium diversity.

An important difference between the attention economy and the information economy is that in the information economy more information per consumer (higher $\phi_2 n_2$) implies a more competitive environment for firms which leads to lower prices (see (2.21)) and is beneficial for consumers.[28] In the attention economy the equilibrium price is independent of the per consumer amount of information. The conventional conjecture of the theory of informative advertising is, that a better average amount of information per consumer should reduces the market power of firms. This is satisfied in the information economy as there is a negative relationship between equilibrium price and per consumer information exposure (see (2.21)). Suppose the goal of a regulating authority were to reduce market power of firms. A policy implication for the information economy then could be, for example, to foster the incentives of firms to advertise by partly subsidising advertising costs $A(\phi)$. This induces more competition and decreases prices. To see this suppose the government decides to introduce a subsidy[29] on advertising: $A(\phi) = \theta(1-\tau)r\Delta m(\phi)$. Without the subsidy we have $\tau = 0$. The introduction of the subsidy reduces the advertising expenditure $\theta r\Delta m(\phi)$ at the firm level for given ϕ. In the information economy the introduction of the subsidy increases equilibrium advertising per firm and also average information exposure.[30] Figure 2.4 illustrates the consequences of this subsidy for the attention and the information economy As the figure shows the subsidy has the effect of increasing equilibrium information per consumer in both economies. In the information economy (the dashed line) we have $y_2 = y(t, \Delta, F, \theta, r, a, c)$ and, because higher average exposure to information decreases prices, the equilibrium price decreases under the subsidy. However, the price locus of the attention economy (the solid line) is flat because we have $y_1 = y(t, c, R)$. Hence the policy measure inferred from the information economy, or more generally from the conventional theory of informative advertising, would not achieve its goal of reducing markups under limited attention.[31] This example suggests that policy measures in an attention economy may have different impacts on the equilibrium than in an information economy. Moreover, the example should make clear that simple measures

[28]This is not a complete welfare statement as consumer utility depends also on how close the firm and the consumer are located apart. I will take into account the consumer's cost of transportation in chapter 4

[29]As this is partial equilibrium analysis I assume that the subsidy is financed e.g. by raising a tax on the nummeraire market (which is not modeled here).

[30]The proposed subsidy formally has the same effect as an exogenous reduction of θ. We have $sign\left(\phi'\left(\theta\right)\right) = sign\left(A_\theta\left(\phi,\theta\right) - \phi A_{\phi\theta}(\phi,\theta)\right)$. But as $\phi A_{\phi\theta}(\phi,\theta) - A_\theta\left(\phi,\theta\right) = \frac{\phi A_\phi(\phi,\theta) - A(\phi,\theta)}{\theta} = \frac{F}{\theta} > 0$ we get $\phi'(\theta) < 0$. From lemma 2.3 we have $\frac{\partial}{\partial\theta}\Omega^N < 0$.

[31]As we will see in chapter 4 a policy measure derived from the conventional theory, without taking into account the consequences of limited attention, may even produce adverse results in an information-rich economy.

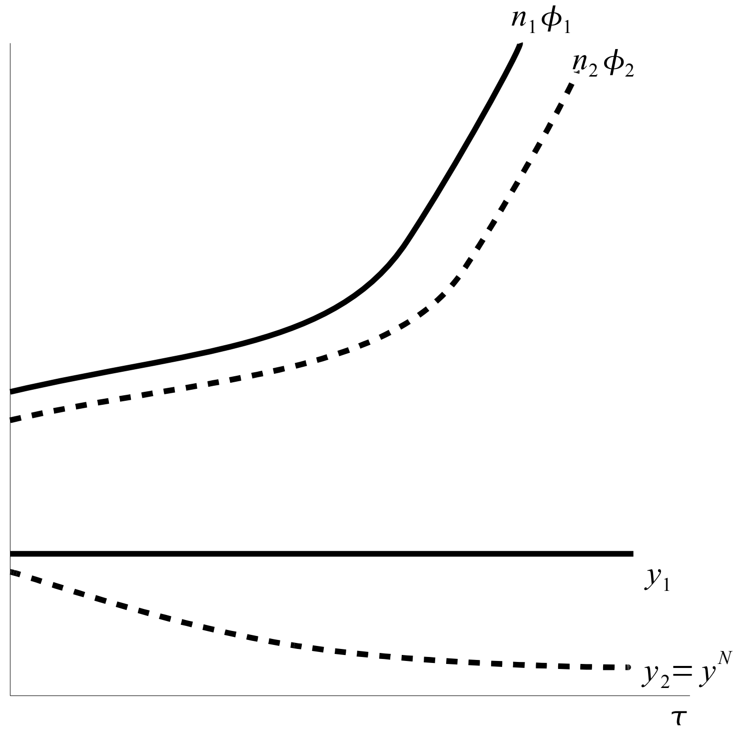

Figure 2.4: The effect of the subsidy in an attention (solid line) and an information economy

such as the number of active firms in a market or information exposure per consumer may be misleading measures of competition if limited attention is not taken into account.

A comparison of my model's predictions to those of Falkinger's attention model (Falkinger (2008)) shows the following. In both models the equilibrium number of firms[32] depends negatively on the reach of a single firm and positively on the size of the potential number of consumers. In my model the reach ϕ is endogenously determined whereas in Falkinger's model the reach is an exogenous characteristic of the firm. Other than in Falkinger's model the attention threshold (R) in my model *negatively* affects the equilibrium n whereas in his model there is a positive relationship between the attention threshold (τ_0) and the measure of active firms. This difference originates from the fact that in the strategic equilibrium of my model the perceived market size directly affects the markup which in turn affects profits.

[32] By imposing that a firm can produce at most one commodity "global diversity" (S) and the measure of active firms (T) coincide (Falkinger (2008), p. 1601).

2.3.3 Advertising technology and equilibrium diversity

From lemma 2.3 we know that advertising technology is an important determinant of whether a conventional or an attention equilibrium occurs. The parameters r and a determine the reach ϕ for a given campaign size m (see (2.5)). From (2.29) we know that higher r and higher a rather imply that an attention equilibrium occurs. In this section I discuss the comparative-static effects of changes in a and r in greater detail. Let $A(\phi, \alpha) = \theta r \Delta m(\phi, r, a)$ where $\alpha = r$ or a. In the appendix (see 2.5.15) I show that

$$sign\left(\phi'(\alpha)\right) = sign\left(A_\alpha(\phi, \alpha)\left(1-\beta\right)\right) \tag{2.30}$$

and

$$sign\left(n'(\alpha)\right) = sign\left(A_\alpha(\phi, \alpha)\left(\xi(\beta-1)-\varepsilon\right)\right) \tag{2.31}$$

where $\beta \equiv \frac{A_{\phi\alpha}\phi}{A_\alpha}$, $\varepsilon \equiv \frac{A_{\phi\phi}\phi}{A_\phi} > 0$ and

$$\xi = \begin{cases} 1 & R < \Omega^N \\ 2 & R > \Omega^N \end{cases}$$

We have $A_r, A_a < 0$ (see 2.5.13 in the appendix.). An exogenous increase of r or an exogenous increase of targeting abilities (an increase of a) ceteris paribus reduce the cost of maintaing a certain reach ϕ. However, the equilibrium effects of such changes on ϕ and n are not obvious. Suppose for a moment that α represents some general parameter of a cost function $A(\phi, \alpha)$. Assume that $A_\alpha < 0$. Then if α increases advertising costs decrease and profits increase (ceteris paribus). Thus we might expect that a higher α leads to entry and more active firms in equilibrium. Expression (2.31) shows that this conclusion may be incorrect: if $\beta >> 1$ we have $\phi'(\alpha) > 0$ but $n'(\alpha) < 0$ is possible.[33] The reason for this are strategic effects: if the reduction of advertising costs induces the strategically behaving firms to increase their advertising effort by a large amount this may lead to higher equilibrium advertising expenditure per firm despite the exogenous cost reduction. In such a case profits decline and fewer firms survive in the market. If we compare the consequences of a change of advertising costs in an attention economy to

[33] From (2.30) - (2.31) we see that we can also have $\phi'(\alpha) \leq 0$ and $n'(\alpha) > 0$ but we can never have $\phi'(\alpha) \leq 0$ and $n'(\alpha) < 0$.

an information economy[34] we see that $n'(\alpha) < 0$ is less likely to occur in the attention economy than in the information economy because in the attention economy we have $\xi = 1$. Intuitively, this must be true because in the attention economy the markups are independent of information per consumer which reflects that strategic effects are of less importance as firms are less exposed to competition.

I now investigate how n depends on α in the cases where $\alpha = r$ or a. All calculations are in the appendix. Let $\hat{A}(\phi, r)$ denote the CRIR-technology, i.e. the case where $a = 1-r$ (see corollary 2.1). We have $\hat{A}_r(\phi, r) < 0$ and $\beta - 1 - \varepsilon < 0$ as well as $2(\beta - 1) - \varepsilon < 0$. Hence we get $n'(r) > 0$ in the CRIR case. If however r and a are independent then we have $n'(r) < 0$ as well as $n'(a) < 0$.[35] This also implies that $\phi'(r) > 0$ and $\phi'(a) > 0$.

As was suggested in section 2.1.2 in a modern economy it is reasonable to assume better targeting abilities of firms (higher a) and hence lower costs of maintaing a certain reach ϕ as well as higher per channel reach r. Numerical evaluations show that the negative effects of a and r on n are much weaker in case of limited attention as is suggested by figure 2.5: The figure compares equilibrium diversity as a function of a (left) and r

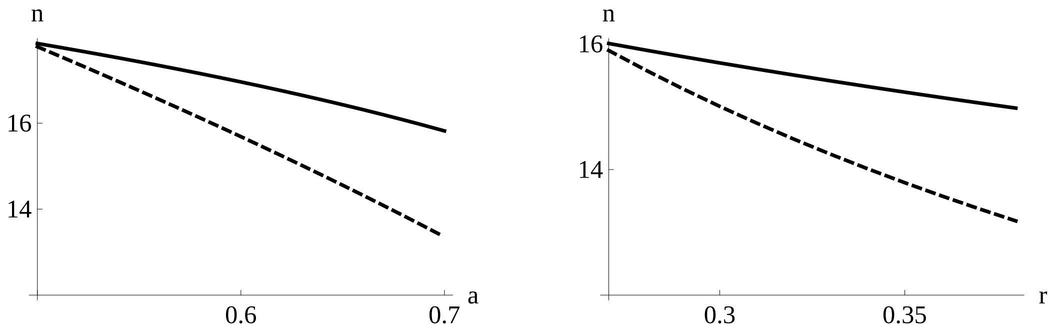

Figure 2.5: $n(a)$ and $n(r)$ under limited (solid) and unlimited attention

(right) of two economies with identical parameter values except for R. The reason why in an attention economy equilibrium diversity decreases by less than in an information economy if target abilities of firms are increased (or r is increased) is the presence of an additional negative equilibrium effect in case of the information economy. In the information economy markups and thus profits depend negatively on average information

[34]As in the last section the two economies are alike up to the value of R.
[35]The general possibility that imposing higher costs of advertising on firms might lead to *higher* profits (or more active firms) was already recognised by GS. The advertising technology of this chapter presents an actual example were this is the case.

$n\phi$ whereas in the attention economy profits are independent of the average information and depend only on the scarcity of attention. Hence in the attention economy profits respond less to an increase of either r or a which explains why equilibrium diversity decreases by less.

2.4 Conclusion

Based on the model of circular product differentiation from GS I have developed a model of informative advertising and limited attention with strategically behaving firms. Limited attention means that consumers are only capable of considering at most R distinct alternatives. If on average the consumers receive information of more than R alternatives an attention equilibrium occurs. This is more likely to happen if advertising is cheap or technologically more efficient (such that more consumers can be reached), firms have better targeting abilities or the varieties are less substitutable. For strategically behaving firms only the market as perceived by consumers is relevant for their pricing and advertising decision. In an information economy the average perceived market size is determined by average exposure of consumers to information ($n\phi$). In an attention economy the average average perceived market size is determined by the attention threshold R. In my model this means that in the attention economy the pricing decision depends on R and not on average per-consumer information $n\phi$ which implies that the pro-competitive potential of informative advertising is severely reduced under limited attention and effective market size n or exposure of consumers to information can be very misleading measures of competition.

At the moment two important issues remain unsolved. First, this model allows for welfare evaluation and GS show that the average transportation costs of the consumers decreases if they receive more information. I will address the consequences of limited attention on average transportation costs in chapter 4. Also the fact that attention-constrained consumers decide randomly among the information they receive seems unsatisfactory as for example an ad posted on the front page of a newspaper or a TV-commercial at the prime time might lead to stronger representation of the information on the consumer's mind which might increase the chance of making a sale to that consumer. Hence we need a formal way of describing this competition for a consumer's attention. It is the task of the next chapter to provide an analytically well tractable and intuitive formulation of how strategic attention competition interacts with strategic economic competition.

2.5 Appendix

2.5.1 Proof of proposition 2.1

Because of $P(i|\, j) = 1 - a_j$ we also have $P(\neg i\, |j) = a_j$, where $\neg i$ means that the consumer does not attend to channel i. Hence $|\delta(j \backslash i)| = a_j \delta$. Because of 2.1 iii) we can write

$$|\delta(k \backslash (i \vee j))| = P(\neg i \wedge \neg j|\, k)\, \delta = P(\neg i|\, k)\, P(\neg j|\, k)\, \delta = a_j^2 \delta$$

or generally for any $i \in \mathcal{M}$ and any $M_j \in \mathcal{M}$ with $i \notin M_j$ and $|M_j| = k$

$$|\delta(i \backslash M_j)| = a_j^k \delta$$

Using (2.2) we get for $a_j < 1$:

$$\Phi(M_j) = \delta \sum_{k=0}^{m_j - 1} a_j^k = \delta \frac{1 - a_j^{m_j}}{1 - a_j}$$

and for $a_j = 1$: $\Phi(M_j) = m_j \delta$

$\blacksquare$

2.5.2 The sign of Φ_a and ϕ_{ma}

Set $a_j = a \in (0, 1)$ and assume $m > 1$. Then differention of (2.4) yields

$$\frac{\partial}{\partial a} \Phi = \frac{\delta\,(a + a^m\,(a(m-1) - m))}{(1-a)^2 a}$$

Then we have $\frac{\partial}{\partial a}\Phi > 0$ if $\psi(a) = 1 + a^m(m-1) - a^{m-1}m > 0$. But $\psi(0) = 1$, $\psi(1) = 0$ and $\psi'(a) = (m-1)ma^{m-2}(a-1) < 0$ together imply that $\psi(a) > 0$ for $a \in (0, 1)$ and $m > 1$.

We have

$$\phi_{ma} = \frac{a^{-1+m} r\,(a - 1 + (a(m-1) - m)\, Ln(a))}{(1-a)^2}$$

Hence

$$\begin{aligned} \phi_{ma} > 0 \quad &\Leftrightarrow (m(a-1)-a)\, Ln(a) > 1-a \\ &\Leftrightarrow -aLn(a) > (1-a)\,(1+Ln(a)m) \end{aligned}$$

The right side of the last inequality decreases in m because $Ln(a) < 0$ as $a \in (0,1)$. Hence it suffices to prove the inequality for $m = 1$. Then

$$-aLn(a) > (1-a)\,(1+Ln(a)) \quad \Leftrightarrow \quad -Ln(a) > 1-a \quad \Leftrightarrow \quad ae^{1-a} < 1$$

which is satisfied because ae^{1-a} increases in a for $a \in (0,1)$ and $\lim_{a\to 1} ae^{1-a} = 1$.

2.5.3 Derivation of (2.7)

Suppose $n > R$ and the representative firm succeeds in informing a member of group k. Suppose further that this member also receives information of $z \in \{0, 1, ..., n-1\}$ other firms. In order to calculate $E\left[q_k, R\,|j \in I_k\right]$ I separate the cases where $z < R$ (limited information) and $z \geq R$ (limited attention).

Case 1: $z < R$

If $z < R$ then the consumer perceives all $z+1$ firms[36] in his information set. Hence the representative firm gets this consumers' demand ($q_k = 1$) only if the z other firms in the information set are inferior[37] compared to the representative firm. For given n and k there are $(n-k)$ inferior firms. For example, let $n = 10$ and consider the consumers in group $k = 3$. Then there are two firms that, compared to the representative firm, provide a superior offer to these consumers. There are 7 firms that provide an inferior offer. Hence the representative firm makes a sale to this consumer if the consumer only receives ads of the inferior firms. Obviously, if $R > z > n-k$ then $q_k = 0$ because there must be at least one superior firm in the information set of the consumer which is perceived (as $R > z$). Because channel selection is random and all competitors choose $\bar{\phi}$, the probability that the consumer is informed of z particular other firms is $\bar{\phi}^z(1-\bar{\phi})^{n-1-z}$. Then

$$P_k(S(z)\,|z < R, j \in I_k) = \bar{\phi}^z(1-\bar{\phi})^{n-1-z} \binom{n-k}{z} \tag{2.32}$$

[36] Remember that I have assumed that the representative firm has informed this consumer.

[37] "Inferior" and "superior" are relative to firm j, the representative firm, and are always measured in net utility (for given prices y and $\bar{y}$).

is the conditional probability that firm j realises a sale to a member of group k given $j \in I_k$ and the consumer also received information of $z < R$ competitors.[38] To see (2.32) let $n = 10$, $R = 5$ and $k = 3$. Suppose for the sake of illustration that firms g and h are the two firms that provide superior offers to a member of group $k = 3$ if compared to the representative firm. If $z = 0$, so that the consumer only has information about the representative firm, then the probability to make a sale to a member of group $k = 3$ is $(1 - \bar{\phi})^9 \cdot 1$: conditional on informing the consumer, the firm always makes a sale to this consumer if no other firm reaches him (i.e. $z = 0$). This occurs with probability $(1-\bar{\phi})^9 \cdot 1$. If $z = 1$ this means that one competitor also managed to inform the consumer and the other $n-2$ competitors have not. The probability for this to occur is $\bar{\phi}(1-\bar{\phi})^{n-2}$. In this case the representative firm makes a sale only if the other firm is not either g or h but one of the $\left(\begin{array}{c} 7 \\ 1 \end{array}\right) = 7$ inferior firms. Generally, there are $\left(\begin{array}{c} n-k \\ z \end{array}\right)$ possibilities to draw a sample of z firms out of the $n - k$ inferior firms.

Case 2: $z \geq R$

Now assume $z \geq R$. Then the firm gets $q_k = 1$ if it is perceived and no superior firm is perceived. The calculation of the sale probability for a given $z \geq R$ naturally becomes more involved as it depends on the probability that a consumer receives information about exactly z competitors as before but also on the probability of being the superior firm among all R *perceived* firms. Different than in the case of limited information the representative firm may be able to realise a sale even if the consumer received information about superior firms. For this to happen we first require $j \in \tilde{A}_k \subset I_k$, which means that the firm needs to be among the R perceived alternatives. Second, if the consumer receives information of superior firms then $q_k = 1$ only if the consumer overlooks the informations of all superior firms. In section 2.5.4 I show that if the subset $\tilde{A}_k$ of perceived alternatives

[38] Note that $z > n - k$ implies $\left(\begin{array}{c} n-k \\ z \end{array}\right) = 0$ by definition of the binomial coefficient (also see section 2.5.4 of the appendix).

is randomly selected from I_k then

$$P_k(S(z)\,|z \geq R, j \in I_k) = \sum_{s=0}^{k-1} \bar{\phi}^z(1-\bar{\phi})^{n-1-z} \begin{pmatrix} k-1 \\ s \end{pmatrix} \begin{pmatrix} n-k \\ z-s \end{pmatrix} \frac{\begin{pmatrix} z-s \\ R-1 \end{pmatrix}}{\begin{pmatrix} 1+z \\ R \end{pmatrix}} \tag{2.33}$$

where the index $s = 1, ..., k-1$ counts the number of superior firms for a given group k. $P_k(S(z)\,|z \geq R, j \in I_k)$ is the conditional probability that firm j realises a sale to a member of group k given $j \in I_k$ and the consumer received information of $z \geq R$ competitors.

We can now calculate the firm's expected conditional demand from a member of group k for any given $\bar{\phi} > 0$ and $n > R$ by taking the sum over the two cases:

$$E\,[q_k, R\,|j \in I_k] = \sum_{z=0}^{R-1} P_k(S(z)\,|z < R, j \in I_k) + \sum_{z=R}^{n-1} P_k(S(z)\,|z \geq R, j \in I_k)$$

or

$$\begin{aligned} E\,[q_k, R\,|j \in I_k] &= \sum_{z=0}^{R-1} \bar{\phi}^z(1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix} \\ &+ \sum_{z=R}^{n-1} \sum_{s=0}^{k-1} \bar{\phi}^z(1-\bar{\phi})^{n-1-z} \begin{pmatrix} k-1 \\ s \end{pmatrix} \begin{pmatrix} n-k \\ z-s \end{pmatrix} \frac{\begin{pmatrix} z-s \\ R-1 \end{pmatrix}}{\begin{pmatrix} 1+z \\ R \end{pmatrix}} \end{aligned} \tag{2.34}$$

So far I have only considered the case where $n > R$. If $R \geq n$, i.e. the number of active firms is smaller than the attention threshold R, then limited attention does not affect $E\,[q_k, R\,|j \in I_k]$ because even a consumer who receives information of all firms can completely process this information. In this case the representative firm can only make a sale to a member of group k if this consumer has not received any information of one of the $k-1$ superior firms. For any $R' > R \geq n$ we have

$$\begin{aligned} &E\,[q_k, R\,|j \in I_k] = E\,[q_k, R'\,|j \in I_k] = E\,[q_k, \infty\,|j \in I_k] \\ &= \sum_{z=0}^{n-1} \bar{\phi}^z\,(1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix} = (1-\bar{\phi})^{k-1} \end{aligned} \tag{2.35}$$

See the proof of lemma 2.1 in this appendix for the last step. It is convenient to have one single expression for $E\,[q_k, R\,|j \in I_k]$ that is valid for any $R \gtreqless n$. If $R \geq n$ the sum in B_2

is not defined and I set $B_2 = 0$ in this case. Further $R \geq n$ implies

$$\mathrm{B}_1 = \sum_{z=0}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix} \tag{2.36}$$

because $\begin{pmatrix} n-k \\ z \end{pmatrix} = 0$ if $z \geq n$ by definition of the binomial coefficient. Thus with $B_2 = 0$ and (2.36) we see that (2.7) exactly corresponds to (2.35) if $R \geq n$. This means that we can work with (2.7) independent of $R \gtreqless n$.

2.5.4 Derivation of (2.33)

Regarding the binomial coeffcient I adopt the usual convention that $\begin{pmatrix} a \\ b \end{pmatrix} = 0$ if $b > a$ and $\begin{pmatrix} a \\ b \end{pmatrix} = 0$ if either $a < 0$ or $b < 0$.

Suppose a consumer in group k receives information from the representative firm and also information from $z \geq R$ other firms. Let s denote the number of superior firms (always in net utility) of which the consumer also receives information. Thus for given z, k and s there are $\begin{pmatrix} k-1 \\ s \end{pmatrix}$ possibilities to draw s superior firms out of the total $k-1$ superior firms. The other $z-s$ firms the consumer is informed of then are inferior firms. There are $\begin{pmatrix} n-k \\ z-s \end{pmatrix}$ possibilities to draw exactly $z-s$ inferior firms from the total of $n-k$ inferior firms. Hence for given s, z and k I get a number of $\begin{pmatrix} k-1 \\ s \end{pmatrix} \begin{pmatrix} n-k \\ z-s \end{pmatrix}$ possible combinations. Note that by the above convention this number always is zero if $s > z$ or $n-k < z-s$.

As an example suppose $n = 10$, $k = 4$,$s = 2$ and $z = 4$. Then there totally are $k-1 = 3$ superior firms out of which $s = 2$ have reached the consumer. There are 3 possible ways to combine two out of three superior firms. Further the consumer was reached by $z-s = 2$ out of $n-k = 6$ possible inferior firms. There are 15 possibilities to draw two inferior firms out of six. Thus we get $3 * 15 = 45$ possible combinations.

Suppose that the selection of the $\tilde{A}_k \subset I_k$ perceived items is completely random. Then the representative firm is perceived, i.e. $j \in \tilde{A}_k$, with probability $R/(1+z)$. In order to make a sale the other $R-1$ perceived firms must be inferior. As by presupposition the consumer received information of $z-s$ inferior firms the chance of making a sale then is

$$\frac{R}{1+z}\frac{\begin{pmatrix} z-s \\ R-1 \end{pmatrix}}{\begin{pmatrix} z \\ R-1 \end{pmatrix}} = \frac{\begin{pmatrix} z-s \\ R-1 \end{pmatrix}}{\begin{pmatrix} 1+z \\ R \end{pmatrix}}$$

by Laplace's probability rule. Thus for fixed k, $z \geq R$ we get

$$P_k(S(z)\,|z \geq R) = \sum_{s=0}^{k-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} k-1 \\ s \end{pmatrix} \begin{pmatrix} n-k \\ z-s \end{pmatrix} \frac{\begin{pmatrix} z-s \\ R-1 \end{pmatrix}}{\begin{pmatrix} 1+z \\ R \end{pmatrix}}$$

2.5.5 Proof of lemma 2.1

Suppose that $\bar{y} = y$. Then from (2.6) we have

$$E[Q,R] = \frac{\Delta\phi}{n} \sum_{k=1}^{n} E\left[q_k, R\,|j \in I_k\right]$$

For given $\phi, \bar{\phi}$ and n the expected number of consumers that consume somewhere must be the same for any R under the assumption that every informed person consumes somewhere. Now suppose $E[Q,R] \neq E[Q,R']$. This means that the expected number of consumers served by the representative firm changes as R changes. But because the expected overall number of consumers that consume somewhere is independent of R there exist attention constrained consumer whose *expected* choice of variety must vary with R. This contradicts the assumption that $\tilde{A}_k$ is a random draw from I_k. Hence

$$E[Q,R] = E[Q] = \frac{\Delta\phi}{n} \sum_{k=1}^{n} E\left[q_k, \infty\,|j \in I_k\right]$$

for any given $R > 1$. Using the law of the binomial distribution gives

$$E\left[q_k, \infty \,|j \in I_k\right] = \sum_{z=0}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix}$$
$$= (1-\bar{\phi})^{k-1} \sum_{z=0}^{n-k} \bar{\phi}^z (1-\bar{\phi})^{n-k-z} \begin{pmatrix} n-k \\ z \end{pmatrix} = (1-\bar{\phi})^{k-1}$$

which further implies

$$\sum_{k=1}^{n} E\left[q_k, \infty \,|j \in I_k\right] = \sum_{k=1}^{n} (1-\bar{\phi})^{k-1} = \frac{1-(1-\bar{\phi})^n}{\bar{\phi}}$$

Thus we have $E\left[Q\right] = \frac{\phi \Delta}{n \bar{\phi}} \left(1-(1-\bar{\phi})^n\right)$. The other claims follow as $E[Q, R]$ is linear in ϕ.

■

2.5.6 Derivation of (2.10) and (2.11)

From (2.7) we have that if $k = n$ we must have $B_2 = 0$ and $B_1 = (1-\bar{\phi})^{n-1}$. Hence $E[q_n, R\,|j \in I_n] = (1-\bar{\phi})^{n-1}$ for any $R > 1$. Thus the representative firm can only make a sale to a member of group n if this member receives no other information at all.

I now show that

$$-E[q_1, R\,|j \in I_1] + E[q_n, R\,|j \in I_n] = (1-\bar{\phi})^{n-1} - 1 + \lambda \tag{2.37}$$

From (2.7) we have

$$E[q_1, R\,|j \in I_1] = \sum_{z=0}^{R-1} \left(\bar{\phi}^z (1-\bar{\phi})^{n-z-1} \begin{pmatrix} n-1 \\ z \end{pmatrix} \right) + \sum_{z=R}^{n-1} \left(\bar{\phi}^z (1-\bar{\phi})^{n-z-1} \begin{pmatrix} n-1 \\ z \end{pmatrix} \frac{R}{1+z} \right) \tag{2.38}$$

because

$$\begin{pmatrix} z \\ R-1 \end{pmatrix} \begin{pmatrix} 1+z \\ R \end{pmatrix}^{-1} = \frac{R}{1+z}$$

Use

$$1=\sum_{z=0}^{R-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\right)+\sum_{z=R}^{n-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\right)$$

in (2.38) to find

$$E[q_1,R\,|j\in I_1]=1-\sum_{z=R}^{n-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\right)\left(1-\frac{R}{1+z}\right)$$

With

$$\lambda\equiv\begin{cases}\sum_{z=R}^{n-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\left(1-\frac{R}{1+z}\right)\right)>0 & \text{if } R<n \\ 0 & \text{else}\end{cases}$$

we get

$$-E[q_1,R\,|j\in I_1]+E[q_n,R\,|j\in I_n]=(1-\bar{\phi})^{n-1}-1+\lambda$$

Using (2.37) in (2.9) and rearranging gives (2.10).

I now show that inequality (2.12) is true. If $R\geq n$ the claim is obvious so let $n>R$. Note that λ is ceteris paribus maximal if $\frac{R}{1+z}=0$. Hence

$$\begin{aligned}1-(1-\bar{\phi})^{n-1}-\lambda &\geq 1-(1-\bar{\phi})^{n-1}-\sum_{z=R}^{n-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\right)\\ &=\sum_{z=0}^{R-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\right)-(1-\bar{\phi})^{n-1}\\ &=\sum_{z=1}^{R-1}\left(\bar{\phi}^{z}(1-\bar{\phi})^{n-z-1}\begin{pmatrix} n-1 \\ z \end{pmatrix}\right)+(1-\bar{\phi})^{n-1}-(1-\bar{\phi})^{n-1}>0\end{aligned}$$

As a consequence $\lambda<1-(1-\bar{\phi})^{n-1}$ and also

$$E[q_n,R\,|j\in I_n]-E[q_1,R\,|j\in I_1]<0 \tag{2.39}$$

2.5.7 Second-order conditions for problem (2.8)

From (2.8) it is easy to see that $\Pi_{\phi\phi}=A''(\phi)<0$ for $\phi\in(0,1)$ as well as $\Pi_{yy}=\frac{\phi\Delta}{t}\left(E[q_n,R\,|j\in I_n]-E[q_1,R\,|j\in I_1]\right)<0$ because of (2.39). But $\Pi_{y\phi}=\frac{1}{\phi}\Pi_y=0$ because $\Pi_y=0$ by the first-order condition of y.

2.5.8 Proof of equation (2.14)

For $\bar{\phi} = 1$ (2.11) becomes

$$\lambda = \begin{cases} \frac{n-R}{n} & R < n \\ 0 & else \end{cases} \tag{2.40}$$

Now use (2.40) in (2.10).

2.5.9 Proof of equation (2.15)

Assume $n > R$ and $\bar{\phi} > 0$. Then

$$\lambda = \underbrace{\sum_{z=R}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-1 \\ z \end{pmatrix}}_{A} - R \underbrace{\sum_{z=R}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-1 \\ z \end{pmatrix} \frac{1}{1+z}}_{B} \tag{2.41}$$

But because of the binomial formula

$$\sum_{z=0}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-1 \\ z \end{pmatrix} = 1$$

Hence $(1-\bar{\phi})^n \cong 0$ implies

$$A = 1 - \sum_{z=0}^{R-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-1 \\ z \end{pmatrix} = 1 - (1-\bar{\phi})^n \sum_{z=0}^{R-1} \bar{\phi}^z (1-\bar{\phi})^{-1-z} \begin{pmatrix} n-1 \\ z \end{pmatrix} \cong 1$$

Note that

$$\sum_{z=0}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-1 \\ z \end{pmatrix} \tfrac{1}{1+z} = \tfrac{1}{n} \sum_{z=0}^{n-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n \\ 1+z \end{pmatrix}$$

$$= \tfrac{1}{n\bar{\phi}} \sum_{\zeta=1}^{n} \bar{\phi}^\zeta (1-\bar{\phi})^{n-\zeta} \begin{pmatrix} n \\ \zeta \end{pmatrix} = \tfrac{1}{n\bar{\phi}} \left(1 - (1-\bar{\phi})^n\right)$$

where the last equality follows from $\sum_{\zeta=0}^{n} \bar{\phi}^{\zeta}(1-\bar{\phi})^{n-\zeta}\begin{pmatrix} n \\ \zeta \end{pmatrix} = 1$. Then

$$\begin{aligned} B &= \sum_{z=0}^{n-1} \bar{\phi}^{z}(1-\bar{\phi})^{n-1-z}\begin{pmatrix} n-1 \\ z \end{pmatrix}\tfrac{1}{1+z} - \sum_{z=0}^{R-1} \bar{\phi}^{z}(1-\bar{\phi})^{n-1-z}\begin{pmatrix} n-1 \\ z \end{pmatrix}\tfrac{1}{1+z} \\ &= \tfrac{1}{n\bar{\phi}}\left(1-(1-\bar{\phi})^{n}\right) - (1-\bar{\phi})^{n}\sum_{z=0}^{R-1} \bar{\phi}^{z}(1-\bar{\phi})^{-1-z}\begin{pmatrix} n-1 \\ z \end{pmatrix}\tfrac{1}{1+z} \cong \tfrac{1}{n\bar{\phi}} \end{aligned}$$

Hence

$$\lambda \cong 1 - \frac{R}{n\bar{\phi}} \tag{2.42}$$

2.5.10 The Dorfman-Steiner Theorem

The price elasticity of demand evaluated at $\bar{y} = y$ is given by

$$\begin{aligned} \varepsilon_y &= -\frac{\frac{\phi\Delta}{t}(E[q_1,R|j\in I_1]-E[q_n,R|j\in I_n])\bar{y}}{\frac{\phi\Delta}{n\bar{\phi}}} = -\tfrac{\bar{y}n\bar{\phi}}{t}\left(E\left[q_1,R\,|j\in I_1\right]-E\left[q_n,R\,|j\in I_n\right]\right) \\ &= -\tfrac{\bar{y}n\bar{\phi}}{t}\left(E\left[q_1,R\,|j\in I_1\right]-E\left[q_n,R\,|j\in I_n\right]\right) \end{aligned}$$

Hence approximately (use (2.37) and (2.42) of the appendix and set $(1-\bar{\phi})^{n-1}=0$)

$$\varepsilon_y \cong \begin{cases} \frac{\bar{y}R}{t} & R < n\bar{\phi} \\ \frac{\bar{y}n\bar{\phi}}{t} & R \geq n\bar{\phi} \end{cases} \tag{2.43}$$

Using (2.43) in (2.19) gives (2.20).

2.5.11 Proof of proposition 2.2

Use (2.21) in (2.22) which gives (2.24). By presupposition $\psi(r) > 0$ and $\psi(K) < 0$. Note that $\psi(\phi)$ is continuous on $\phi \in (r,K)$ and $\psi'(\phi) < 0$ for all $\phi \in (r,K)$ except at $\phi = R/n$. Thus we may conclude that there exists a unique $\phi \in (r,K)$ such that $\psi(\phi) = 0$ holds.

■

2.5.12 Proof of lemma 2.2

If equation (2.25) implies that $\phi^N \leq R/n$ then because of (2.24) we must have $\phi = \phi^N$. If we get $\phi^N > R/n$ then ϕ^N cannot be a solution to (2.24). Because $\psi'(\tilde{\phi}) < 0$ for any

$\tilde{\phi} > \phi^N$ (see the proof in 2.5.11) the solution to (2.24) must satisfy $\phi > \phi^N$.

If $n \leq R$ we can never have $\Omega^N > R$ as $\phi^N < 1$.

To see (2.26) suppose x is any parameter in $\mathcal{F}$ other than n or R. Then we have

$$sign\left(\frac{\partial}{\partial x}\Omega^N(x)\right) = sign\left(\frac{\partial}{\partial x}\phi^N(x)\right)$$

Hence by the Implicit Function Theorem we immediately get $\frac{\partial}{\partial t}\phi^N(t) > 0$, $\frac{\partial}{\partial \Delta}\phi^N(\Delta) = 0$, $\frac{\partial}{\partial \theta}\phi^N(\theta) < 0$. For $x = a$ we have $sign\left(\frac{\partial}{\partial a}\phi^N(a)\right) = sign\left(m_{\phi a}(\phi, r, a)\right)$ and section 2.5.2 implies that $\frac{\partial}{\partial a}\phi^N(a) > 0$. For $x = r$ we have $sign\left(\frac{\partial}{\partial r}\phi^N(r)\right) = sign\left(m_\phi(\phi, r, a) + rm_{\phi r}(\phi, r, a)\right)$. But

$$m_\phi(\phi, r, a) + rm_{\phi r}(\phi, r, a) = -\frac{(1-a)^2\phi}{\left(r-(1-a)\phi\right)^2 Ln(a)} > 0$$

which implies $\frac{\partial}{\partial r}\phi^N(r) > 0$. Finally, I use the Implicit Function Theorem on (2.25) to find

$$\frac{\partial}{\partial n}\Omega^N(n) = \frac{\phi^2 A''(\phi)}{2A'(\phi) + \phi A''(\phi)} > 0$$

■

2.5.13 Proof of lemma 2.3

If the solution to (2.28) gives $\Omega^N \leq R$ then $(\phi^N, n^N) = (\phi, n)$ as (ϕ^N, n^N) solves (2.24), (2.27') and the solution of (2.24), (2.27') is unique by presupposition. Thus a conventional equilibrium occurs. If the solution to (2.28) gives $\Omega^N > R$ then (ϕ^N, n^N) cannot be a solution to (2.24), (2.27'). But (2.27') implies $\phi = \phi^N$. Then by (2.24) we must have $n > n^N$ which implies $\phi n > \phi^N n^N > R$; an attention equilibrium occurs. To obtain (2.29) we must derive the comparative statics of (2.28). Let reach costs be $A(\phi, \alpha)$ where $\alpha =\theta$, r or a denotes a parameter of the cost function. The coefficient matrix of system (2.28) with respect to ϕ, n and Ω (the superscript N is dropped to ease notation) is given by

$$H = \begin{pmatrix} \phi A_{\phi\phi} & 0 & 0 \\ -\frac{2t\Delta}{n^2\phi^3} - A_{\phi\phi} & -\frac{2t\Delta}{n^3\phi^2} & 0 \\ -n & -\phi & 1 \end{pmatrix}$$

and $Det(H) = -\frac{2t\Delta A_{\phi\phi}}{n^3\phi} < 0$. Then by Cramer's rule

$$\Omega'(t) = \frac{(-\Delta A_{\phi\phi})}{n^2 Det(H)} > 0$$

$$\Omega'(F) = (\phi A_{\phi\phi}) \frac{1}{Det(H)} < 0$$

Further it is straightforward to show that $sign\,(\Omega_\Delta) = sign\,(\phi A_\phi(\phi) - A(\phi))$. But (2.27') implies $\phi A_\phi(\phi) - A(\phi) = F > 0$ Hence $\Omega'(\Delta) > 0$. In now turn to the cases where $\alpha =\theta$, r or a.

$$\Omega'(\alpha) = (\phi A_\alpha A_{\phi\phi}) \frac{1}{Det(H)}$$

which implies $sign\,(\Omega_\alpha) = sign\,(-A_\alpha(\phi, \alpha))$. Hence $\Omega'(\theta) < 0$ because $A_\theta = r\Delta m(\phi) > 0$. Note that (2.5) implies

$$m\,(\phi, r, a) = \frac{Ln\left(1 - \frac{1-a}{r}\phi\right)}{Ln(a)}$$

We have $sign\,(A_r) = sign\left(m(\phi, r, a) + \frac{\phi(1-a)}{(r-(1-a)\phi)Ln(a)}\right)$ which is equivalent to

$$sign\,(A_r) = sign\left(m(\phi, r, a) + \frac{\phi(1-a)}{(r-(1-a)\phi)Ln(a)}\right)$$

or

$$sign\,(A_r) = sign\left(\frac{r}{r-b} - e^{\frac{b}{r-b}}\right) \qquad b \equiv \phi(1-a) \in (0, r)$$

But then we have $A_r < 0$ because $e^{\frac{b}{r-b}} > \frac{r}{r-b}$. This follows as $b = 0$ implies $e^{\frac{0}{r}} = 1 = \frac{r}{r}$ and

$$\frac{\partial}{\partial b}\left(e^{\frac{b}{r-b}}\right) = e^{\frac{b}{r-b}} \frac{r}{(r-b)^2} > \frac{r}{(r-b)^2} = \frac{\partial}{\partial b}\left(\frac{r}{r-b}\right)$$

Thus $\Omega'(r) > 0$.

We have

$$sign(A_a) = sign\left(\frac{\phi Ln(a)}{r-(1-a)\phi} - \frac{Ln\left(1 - \frac{(1-a)\phi}{r}\right)}{a}\right)$$

This implies $A_a < 0$ as

$$\frac{\phi Ln(a)}{r-(1-a)\phi} < \frac{Ln\left(1 - \frac{(1-a)\phi}{r}\right)}{a}$$

To see this rewrite the last inequality as

$$e^{\frac{\phi a}{r-(1-a)\phi}} > \frac{1-a}{r}(r-\phi)$$

This inequality is satisfied as $\phi > r$ by presupposition of lemma 2.3. Hence $\Omega'(a) > 0$.

■

2.5.14 Proof of proposition 2.3

From (2.27') we have $\Pi'(\phi) = \phi A''(\phi) > 0$ Together with $\Pi(r) < 0$ and $\Pi(K) > 0$ this implies that a unique solution $\phi \in (r, K)$ to (2.27') exists. Moreover, ϕ is independent of the regime type (see lemma 2.3). Hence $\phi'(R) = 0$ for all $R > 1$. With ϕ determined by (2.27') we see that n and y are uniquely determined by (2.24) (also see figure 2.3) and (2.21). Now suppose that parameters in $\mathcal{B}$ are such that an attention equilibrium occurs endogenously. Then $y'(R) < 0$ and $n'(R) < 0$ follow directly from (2.21) and (2.24).

■

2.5.15 Comparative statics effects of r and a

Suppose a solution to (2.24) and (2.27') exists and $\Omega^N \neq R$. The coefficient matrix $\tilde{H}$ of (2.24) and (2.27') with respect to (ϕ, n) is

$$\tilde{H} = \begin{pmatrix} \phi A_{\phi\phi} & 0 \\ -\left(\frac{\xi A_\phi}{\phi} - A_{\phi\phi}\right) & -\frac{\xi A_\phi}{n} \end{pmatrix}$$

where

$$\xi = \begin{cases} 1 & R < \Omega^N \\ 2 & R > \Omega^N \end{cases}$$

Further let $\beta \equiv \frac{A_{\phi\alpha}\phi}{A_\alpha}$ and $\varepsilon \equiv \frac{A_{\phi\phi}\phi}{A_\phi}$. Applying Cramer's rule gives, after some simple manipulations,

$$\begin{aligned} sign\left(\phi'(\alpha)\right) &= sign\left(A_\alpha(\phi,\alpha)\left(1-\beta\right)\right) \\ sign\left(n'(\alpha)\right) &= sign\left(A_\alpha(\phi,\alpha)\left(\xi(\beta-1)-\varepsilon\right)\right) \end{aligned}$$

In the CRIR case ($\alpha = r$) we have

$$\hat{A}_r(\phi, r) = \frac{\Delta\theta\left(r + (1-r)Ln(1-r)\right)Ln(1-\phi)}{(1-r)\left(Ln(1-r)\right)^2}$$

To see that $\hat{A}_r(\phi, r) < 0$ it is sufficient to show that $g(r) = (r + (1-r)Ln(1-r)) > 0$ for $r \in (0,1)$. But $g(0) = 0$, $\lim\limits_{r \to 1^-} g(r) = 1$ and $g'(r) = -Ln(1-r) > 0$ together imply that $g(r) \in (0,1)$ for $r \in (0,1)$. Further, the CRIR technology implies

$$(\beta - 1) - \varepsilon = -\frac{1 + \frac{\phi}{Ln(1-\phi)}}{1-\phi}$$

We get $(\beta-1)-\varepsilon < 0$ as $g(\phi) = 1 + \frac{\phi}{Ln(1-\phi)} > 0$. This holds as $g(0) = 1$, $\lim\limits_{\phi \to 1} g(\phi) = -\infty$ and $g'(\phi) = 1 - \frac{1}{1-\phi} < 0$. Repeating this argument it is straightforward to show that also $2(\beta - 1) - \varepsilon < 0$.

Now consider the case where r and a are independent parameters. If $\alpha = r$ we get

$$(\beta - 1) - \varepsilon = -\frac{(1-a)\phi + rLn\left(1 - \frac{(1-a)\phi}{r}\right)}{\phi(1-a) + (r - (1-a)\phi)Ln\left(1 - \frac{(1-a)\phi}{r}\right)} = -\frac{1 - a^m + Ln(a^m)}{1 + a^m(Ln(a^m) - 1)}$$

where the second equality uses $\phi = r\frac{1-a^m}{1-a}$. Let $x = a^m \in (0,1)$. Then

$$-\frac{1 - a^m + Ln(a^m)}{1 + a^m(Ln(a^m) - 1)} = -\frac{g_1(x)}{g_2(x)} = -\frac{1 - x + Ln(x)}{1 + x(Ln(x) - 1)} \tag{2.44}$$

But forming limites shows that $g_1(0) = -\infty$, $g_1(1) = 0$ and $g_1'(x) > 0$ which implies $g_1(x) < 0$ for $x \in (0,1)$. Similarly, $g_2(0) = 1$, $g_2(1) = 0$ and $g_2'(x) < 0$ implies $g_2(x) > 0$ for $x \in (0,1)$. Consequently, we have $(\beta - 1) - \varepsilon > 0$ if $\alpha = r$. Repeating this procedure in the case where $\alpha = a$ gives

$$\beta - 1 - \varepsilon = -\frac{(1-a)(a^m - 1 - Ln(a^m))}{a(1-a^m)Ln(a) - (1-a)a^m Ln(a^m)}$$

But

$$(1-a)(a^m - 1 - Ln(a^m)) > 0 \quad \Leftrightarrow \quad e^x > x \qquad x \equiv a^m \in (0,1)$$

which is true and

$$a(1-a^m)Ln(a) - (1-a)a^m Ln(a^m) < 0 \quad \Leftrightarrow \quad a(1-a^m) - (1-a)a^m m > 0$$

To see that $a(1-a^m) - (1-a)a^m m > 0$ take the derivative with respect to m of

$$a(1-a^m) - (1-a)a^m m \tag{2.45}$$

which gives $Ln(a)\,(-a-m+am) - 1 + a$. But $Ln(a)\,(-a-m+am) > 1-a$ holds for $m > 1$. To see this let $m = 1$. Then the inequality reduces to $a - Ln(a) > 1$ which is true. For $m > 1$ the factor is smaller than -1. Hence we only need to verify that $a(1-a^m) - (1-a)a^m m \geq 0$ for $m = 1$. But evaluating (2.45) for $m = 1$ gives $a(1-a) - (1-a)a = 0$ which proves the claim. Hence $(\beta - 1) - \varepsilon > 0$ if $\alpha = a$.

As $\varepsilon > 0$ we have that $(\beta - 1) - \varepsilon > 0$ also implies $2(\beta - 1) - \varepsilon > 0$.

3

Attention Competition

3.1 Introduction

In this chapter I integrate the second key observation from the introduction - the possibility of information senders to influence the chance of perception - into an abstract oligopolistic model of price competition. The chance that an information sender is perceived depends positively on his own effort to attract consumer attention but negatively on the efforts and the number of all other senders. In my model economic competition and attention competition are interdependent: On the one hand the effort to attract consumer attention depends on the value of attention to the firm which depends on the grade of economic competition among all perceived firms. On the other hand attracting attention involves costs which must be covered by the earnings from competition. It is the task of this paper to clarify the interaction between attention competition and economic competition in order to give a prediction on how attention competition influences equilibrium market outcome. Under limited attention the market as perceived by consumers rather than the effective market is relevant for the firms. If consumers are less attentive this means that the perceived market gets smaller which, as in chapter 2, translates into higher equilibrium prices. At the same this intensifies the competition for attention which leads to higher attention costs. I show that if attention competition is relatively inelastic (equilibrium attention efforts do not react significantly to a change of the economic environment) then the gains from consumer inattention outweigh the costs of attracting attention which leads to higher profits and thus to larger markets.

This chapter is organized as follows. In section 3.1.1 I provide an informal description of the model and translate these findings into a formal model in section 3.2. I will restrict myself to the case of symmetric firms. This gives a static symmetric game where all active firms simultaneously and non-cooperatively choose their price and their attention effort. In section 3.3 I show that a symmetric equilibrium of the price-attention game exists and discuss the possibility of multiple symmetric equilibria. I extend the model to a two-stage game where firms must decide whether to enter at the first stage and then play a price-attention game at the second stage. I turn to the comparative statics of the model in section 3.4. In Appendix A I provide further conditions that assert contraction stability and uniqueness of the symmetric equilibrium by ruling out the possibility of asymmetric equilibria.

3.1.1 An informal description of the attention problem

In this section I illustrate the attention problem in a schematic and intuitive way. The main part of this chapter formalises these schematics and derives a game-theoretic model where firms (information senders) compete for consumer attention and consumer budget.

3.1.1.1 A simple consumer choice problem

Suppose for the sake of illustration that there are $n = 4$ firms. Each firm produces one commodity and sets a price y_j. Consumer i decides how much to purchase from each commodity. φ_i is the choice function of consumer i. How much of which commodity the consumer purchases (x_{ij}) depends on his preferences and budget v and on the prices of the commodities. This standard decision problem is illustrated in figure 3.1.

3.1.1.2 Informative advertising and limited information

As we have seen in the last chapter the theory of informative advertising holds that consumers are ex ante uninformed about existing commodities. They learn about the commodities only by the information (ads) they receive from the firms and choose only among the commodities they are aware of. Such a situation is depicted in figure 3.2.

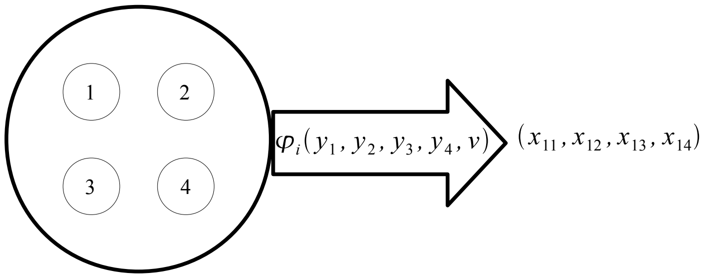

Figure 3.1: A classical consumer decision problem

In the figure we see that only firms two and four managed to inform consumer i. That

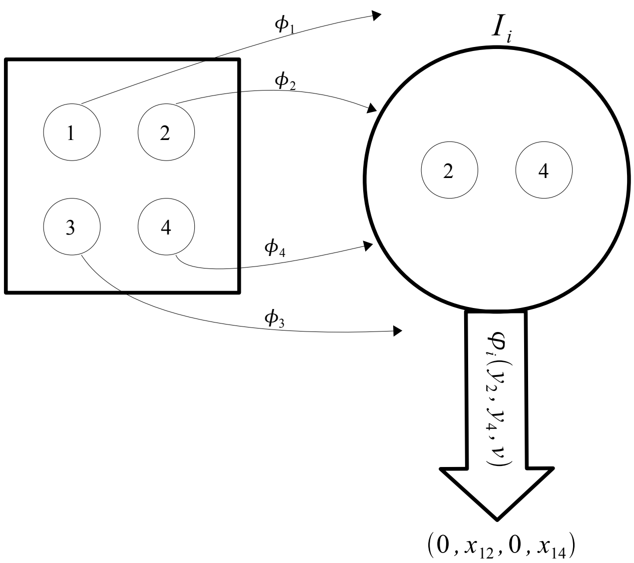

Figure 3.2: Limited information

is, the consumer received ads only from these firms but not of firm one and three and he decides only among those firms he is informed of. If the firms can choose how many consumers to inform (for example if they can decide in how many different newspaper to put an ad) then the information set I_i is endogenously determined and depends on the advertising efforts of the firms. In the figure this is suggested by the firm-specific variable ϕ_j. The literature on informative advertising says that a consumer who considers a set of alternatives that is smaller than the set of all existing alternatives has *limited information* (see e.g. Goeree (2008)).

3.1.1.3 Scarce attention

In the last chapter I presented a model of limited consumer attention. Limited attention means that consumers are only capable of perceiving a certain amount R of alternatives. Suppose for the sake of illustration that $R = 2$. If a consumer receives information of not more than two firms then he considers all alternatives he is informed about in making his decision. Such a situation is captured by figure 3.2. Now suppose consumer i received information from firms one, two and four but not from firm three, as depicted in figure 3.3. Limited attention means the consumer bases his decision on a subset of I_i that

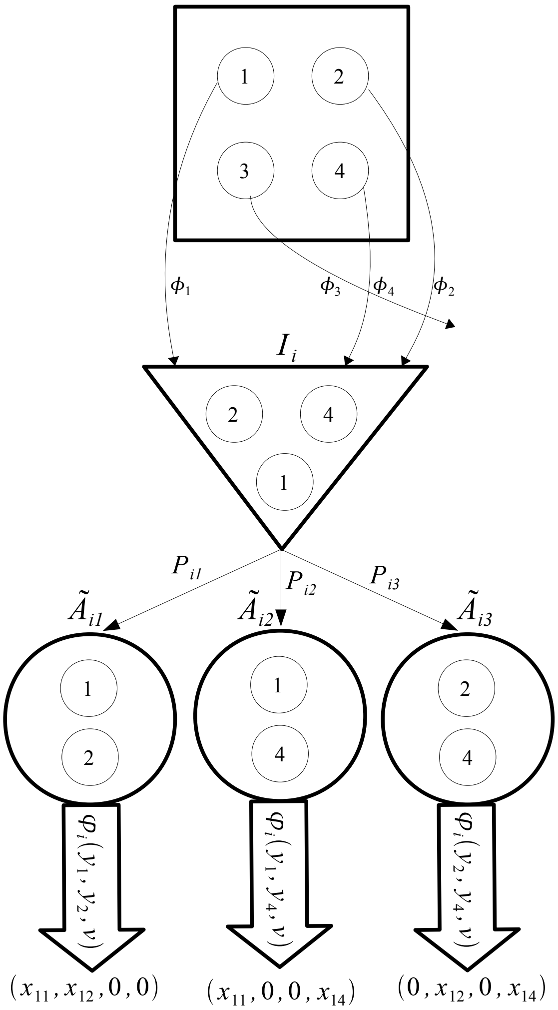

Figure 3.3: Limited attention

contains $R = 2$ different alternatives. As figure 3.3 suggests there are three possibilities to pick two alternatives out of the set of three alternatives and $P_{i\alpha}$ for $\alpha = 1, 2, 3$ with

$P_{i1} + P_{i2} + P_{i3} = 1$ denotes the probability with which consumer i observes a certain attention set $\tilde{A}_{i\alpha}$. In this chapter I will argue, based on the evidence of the introduction (see 1.2), that the firms which are a part of I_i have means to influence the probability distribution (P_{i1}, P_{i2}, P_{i3}). That is, firms can influence their *chance of perception*. I think that this properly reflects the way how people who use the internet, especially search engines, make their decision: they are presented with a large list of alternatives and consider (click on) only a few alternatives - especially those alternatives that have top positions or are highlighted by some other means (e.g. in the Google sideframe).

Note that by the notion of the literature on informative advertising the consumer in figure 3.3 also has limited information - but for a very different reason: the cause is not that he received only little information but rather that he received more information than he considers. My contribution essentially shows that if limited information is caused by limited attention rather than scarce information this generates a very different strategic environment and implies very different results compared to standard models of informative advertising.

As figure 3.3 suggests in order to earn money from a consumer a firm must achieve three goals: i) the firm must inform the consumer (e.g. receive an index by a Google search bot), ii) it must be perceived by a consumer (e.g. because it has a high index and a top on-screen placement) and iii) it must provide a satisfying offer in a conventional economic sense compared to all other firms that the consumer perceives[1]. One contribution of this chapter is to provide a specification of how the firms in I_i can influence their chance of perception. This means that the probability distribution (P_{i1}, P_{i2}, P_{i3}) in figure 3.3 will be endogenously determined and depends on the strategic attention efforts of the firms in I_i. What I want the model to capture is that if in figure 3.3 firm one's messages are *relatively conspicuous* compared to the messages of firms two and four then we should have $P_{i1} + P_{i2} \approx 1$.

To make the difference between figure 3.2 and figure 3.3 as clear as possible suppose firms can advertise only by posting ads in different newspapers. Consumers do not read all newspapers. If a firm places its ads in more newspapers this increases the number of

[1]Suppose that consumer i perceives the set $\tilde{A}_{i1}$ in figure 3.3. Then firm one competes with firm two for the consumer budget v.

people that could see the ad. Put differently, this increases the number of information sets that contain the firm. Suppose now a consumer reads a newspaper that contains many ads of different firms. Then it is reasonable to assume that a large or colorful ad or an ad placed on the front page has a larger chance of getting the readers attention than a small ad somewhere on the last pages. That is, contrary to the theory of informative advertising, a firm that advertises in many newspapers but has very non-salient ads may go unnoticed by most consumers. In essence, it is this struggle for attention which is the main concern of this chapter. This is also the major difference to informative advertising as discussed in chapter 2. In that chapter more advertising meant to inform more consumers (to reach more information sets). More advertising of the competitors made consumers aware of more alternatives which diminished demand of the representative firm. Limited consumer attention implied this reduction to be a lot weaker than suggested by the conventional model. In the model of this chapter higher effort to attract attention will mean that the firm increases its chance of perception for a given number of consumers. This means that the information competition between the firms in this chapter is a lot more aggressive in its nature than in the last chapter because if a firm chooses a higher attention effort it increases its own chance of perception while at the same time *inhibiting* perception of some other firms.

3.2 A formal model of attention and price competition

3.2.1 The Internet economy

As was shown in the introduction of this thesis the usage of the internet has rapidly expanded since its introduction (see figure 1.5). For many people the internet nowadays is the main source of information and, as e.g. the stories of Google's or facebook's success document, the internet also offers great business opportunities. But the internet, especially a search page such as Google, also provides the most convincing example of the prevalence of limited attention. A typical Google search query usually results in an enormous number of hits[2]. As suggested by the observations documented in the introduction of my thesis, the typical user of a search engine focuses on the first few entries (or on the first few pages) instead of threading through the entire list. Moreover, this information is accessible for any consumer in the world provided he has access to the internet. This clearly illustrates that the contemporaneous (and future) information problem of an advertising firm is less one of reaching more consumers (more information sets) but one of making its messages salient. In the Google example this e.g. means to be listed on the first page of a search query.[3] In this chapter I extrapolate this observation by assuming that every active firm that pays a fixed cost $F > 0$ reaches the entire population Δ. Hence all consumers have the same information set: $I_i = I$.[4] In the Google example this means that if a firm designs a web page and puts it online (achieving this requires some fixed amount of money) it is found and indexed by a Google bot. However, whether the firm has a high or a low index (a top rank position or not) depends on further investments[5] of the firm. Setting $I_i = I$ appears also to be adequate for mature product classes. For such products, e.g. Coke and Pepsi, the role of advertising is not information provision about the features but "keeping a product top-of-mind" (Iyer et al. (2005), p. 464) for which "these companies spend a significant amount of their budget on reminder-advertising". I simply call an economy where a firm is either in all or in no information set an "internet economy". If a firm pays the setup cost F then I call the firm active. Suppose there are n active firms indexed by $1, ..., n$ in the internet economy. Then $n = |I|$.

3.2.2 Attention and price competition in the internet economy

In this section I formalise the schematics of figure 3.3. First, I develop a model that determines how the attention efforts of the active firms influence their probability of being perceived by a consumer. Then I combine this model with a model of imperfect price

[2] For example, the keyword "Hawaii" gives approximately 123'000'000 hits and the query "Hawaii vacations" gives approximately 8'160'000 hits (January 09, 2011).

[3] As a Web commentator puts it: "Simply, you want to be found at the common meeting point...page 1 of a Google search" (see www.optimum7.com).

[4] In the model of Falkinger this corresponds to the case where $r = R$, i.e. a sender reaches the entire population (Falkinger (2008)). In my model of chapter 2 if $r = 1$ we get $\phi_j = 1$ for any firm j that acquires one channel.

[5] For example, the firm can try to get more cross-links to its webpage or can bid for a placement in the Google sideframe. In both cases there is some kind of variable cost involved.

competition. I will provide a simple example that accompanies and illustrates the abstract theory developed in this section. In the end of section 3.2 we will have a completely specified model of n active firms that simultaneously and non-cooperatively choose their attention effort and their price. I will restrict myself to the case of symmetric firms. Thus formally we will have to deal with a static two-dimensional symmetric n-player game.

3.2.2.1 Attention probabilities

Suppose there are n active firms in the internet economy and hence $n = |I|$. The power set of I is denoted by $\mathbb{P} \equiv \mathbb{P}(I)$ and I call an element $A \in \mathbb{P}$ an attention set. As in chapter 2 limited attention means that only attention sets not larger than some $1 < R < \infty$ are considered by a consumer[6]. I denote the attention set of consumer i (the set of alternatives that the consumer effectively perceives) by $\tilde{A}_i \subset I$. Limited attention means that $\left|\tilde{A}_i\right| \leq R$. The attention constraints of the consumers are strictly binding if and only if $n > R$. If $n \leq R$ then we have $\tilde{A}_i = I$ for all consumers in the internet economy. If however $n > R$ then we have $\tilde{A}_i \subsetneq I$ and $\tilde{A}_i \neq \tilde{A}_h$ is possible for any two consumers. In case of a binding attention constraint literally spoken only some commodities can make it into the attention set $\tilde{A}_i$ of a consumer and the next question is which alternatives succeed in doing so. To answer this question we need to specify a rule that for a given I tells whether $j \in I$ implies $j \in \tilde{A}_i$ or not. The general way to map a larger set into a smaller one is to specify a probability distribution on the set of all possible attention sets. Define the indicator variable z by

$$z = \min\{R, n\} \tag{3.1}$$

Let $\mathcal{A}(I, z) \subset \mathbb{P}$ be the set of all subsets of I with size z.[7] For every consumer i I now define a function P_i on $\mathcal{A}$ that assigns to every possible attention set A of $\mathcal{A}$ with size z a probability[8] of realising this particular set:

$$P_i : \mathcal{A} \to s^{|\mathcal{A}|} \quad , A \mapsto P_{iA} \tag{3.2}$$

[6] As in chapter 3.3 I let $R_i = R$ for all consumers.

[7] Because by definition of the internet economy we have $I_i = I$ for all consumers the set $\mathcal{A}$ of possible attention sets is the same for all consumers. This would not be the case if we allowed for $I_i \neq I_h$.

[8] Of course, the probability of a certain set could also be one ore zero.

where $s^{|\mathcal{A}|}$ is the $(|\mathcal{A}|-1)$-dimensional simplex[9] and $A \in \mathcal{A}$ is an attention set of size z. Thus P_{iA} is the probability that $\tilde{A}_i = A$, i.e. P_{iA} is the probability that consumer i perceives the attention set A. Note that we have $z = n$ if $R \geq n$. But then we must have $\mathcal{A} = I$ and $|\mathcal{A}| = 1$. The only possible function satisfying (3.2) in this case is the unit function which gives $\tilde{A}_i = I$ for all consumers. Hence the entire apparatus I am about to construct only comes into play when we have $n > R$, i.e. when the attention constraints of the consumers bind.

Before continuing it should be noted that by definition of (3.2) I only allow the attention sets to be proper sets (instead of multisets). This can be justified by the assumption that attention-constrained decison-makers cannot be fooled in the sense that they always recognize and consider R truly distinct alternatives.[10]

The probability p_{ij} that firm j is perceived by consumer i given that $j \in I$ is calculated by

$$p_{ij} = \sum_{A \in \mathcal{A}} P_{iA} \, 1 \left[j \in A \right] \tag{3.3}$$

where $1 \left[j \in A \right]$ is a variable indicating whether alternative j is in attention set A or not. The following proposition establishes an aggregate relationship on P_i and p_{ij}.

Proposition 3.1. *If P_i is a probability function as defined by* (3.2) *then* $\sum\limits_{j \in I} p_{ij} = z$.

Proof: Appendix B (3.7.2)

Conversely, it generally only is possible to construct P_i from the set of p_{ij} where $j \in I$ in certain trivial cases.[11] The vector $p_i^A \equiv \left(p_{ij}\right)_{j=1,\dots,n}$ for $j = 1, \dots, n$ can be interpreted as the distribution of attention of consumer i. Likewise, if there are Δ consumers then the n-vector

$$p^A \equiv \frac{\sum\limits_{i=1}^{\Delta} p_i^A}{\Delta}$$

is the average distribution of attention in the market. Then the j-th entry of p^A corresponds to the average fraction of consumers in the economy that consider an attention

[9] $s^{|\mathcal{A}|} \equiv \left\{ P \in \mathbb{R}_+^{|\mathcal{A}|} : \sum\limits_{A \in \mathcal{A}} P_{iA} = 1 \right\}$.

[10] This means that even if a firm j by some reason has a high chance of perception, i.e. $\sum\limits_{A \in \mathcal{A} : j \in A} P_{iA} \approx 1$, then for $R > 1$ this does not mean that it is the only firm that is perceived.

[11] A nice property of the symmetric game is that P_i can be deduced from p_{ij}.

set containing j.

3.2.2.2 Limited attention and economic competition: the problem of the firm

From an economic perspective the function P_i becomes interesting if actions chosen by the market agents such as advertising efforts by firms (or search efforts by consumers) affect the probability distribution. Throughout this chapter I will maintain the assumption that P_i is determined only by the attention efforts of the firms in I. Suppose every active firm j can choose its attention effort $f_j \geq 0$. I denote by f_{-j} the vector of attention efforts chosen by all active firms other than j. Let $\mathcal{F} \equiv (f_1, ..., f_n) \in \mathbb{R}_+^n$ denote the vector of attention efforts. I assume that the attention probabilities P_A depend on $\mathcal{F}$. Further, for any active firm j define the set $B_j \equiv \{A \in \mathcal{A} : j \in A\}$.

Assumption 3.1 (Relative salience)**.** *For all consumers $P_i = P$ where the function P is defined by* (3.2)*. If $\sum\limits_{A \in B_j} P_A < 1$ then $P_A(\mathcal{F})$ is an increasing function of f_j for all $A \in B_j$.*

The first part of assumption 3.1 means that all consumers have the same function P which determines the probability that a particular attention set $A \in \mathcal{A}$ is perceived. This assumption is mainly for simplicity. The second part of assumption 3.1 means that under limited attention $(n > R)$ a firm can positively influence the chance that a consumers perceives an attention set A with $j \in A$.

Lemma 3.1. *Under assumption 3.1 we have for any active firm j and any consumer i that $p_{ij} = p_j$. If $R \geq n$ then $p_j = 1$ for any active firm j. If $n > R$ then $p_j = p_j(f_j, f_{-j})$ where $p_j(f_j, f_{-j})$ is an increasing function of f_j if $p_j < 1$.*

Proof: Appendix B (3.7.3)

To formalise the interdependence of economic competition and attention competition I assume that every firm non-cooperatively chooses one variable y_j, henceforth interpreted as its price.

Let $Y_A \equiv (y_{u1}, ..., y_{uz}) \in \mathbb{R}^z$ with $uk \in A$ denote the vector of prices of those firms that belong to A. For every $A \in \mathcal{A}$ define a function $V^j(Y_A) : \mathbb{R}^z \rightarrow \mathbb{R}^1$ with the property that $V^j(Y_A) = 0$ if $j \notin A$. Then $V^j(Y_A)$ is the value of attention set A to firm j and summarises the value earned by firm j from the economic competition between the firms in attention set A. As an example think of price competition with substitute products and linear production costs. Then

$$V^j(Y_A) = (y_j - c)x_j(Y_A) \tag{3.4}$$

where $x_j(Y_A)$ is the demand function of a consumer facing attention set A with price vector Y_A. The second assumption I impose with the goal of eventually writing down the payoff function of firm j is the following:

Assumption 3.2 (Separability)**.** *For any active firm j and any $A \in B_j$ the function $V^j(Y_A)$ is independent of $(f_1, ..., f_n)$.*

This is a very important assumption. It means that conditional on an attention set consumers make decisions *independent* of all attentional activities.[12] Taken to the Google example the assumption means that the rank of an alternative only influences its chance of perception but does not convey more economically relevant information to the consumer. Formally, this assumption enables me to separate the competition for an attention set from the economic competition within an attention set and leads to an analytically highly tractable structure of such a problem.

Under assumptions 3.1 and 3.2 the expected profit function of firm j is

$$\Pi^j\left((y_1, f_1), ..., (y_j, f_j), ..., (y_n, f_n)\right) = \sum_{A \in B_j} P_A(\mathcal{F})V^j(Y_A)\Delta - F - C(f_j) \tag{3.5}$$

where $C(f_j)$ denotes the cost firm j must incur under effort level f_j. The properties of C will be discussed below. The fixed setup cost $F > 0$ can be thought of as summarizing infrastructure costs for production and IT.

In case of (3.4) the profit function (3.5) becomes

$$\Pi^j = (y_j - c) \sum_{A \in B_j} P_A(\mathcal{F})x_j(A)\Delta - F - C(f_j) \tag{3.6}$$

[12]This is a crucial difference between attention competition as understood by my contribution and persuasive advertising: persuasive advertising would mean that the advertising effort of a firm could influence the function V^j. Attention competition means that the chance of perception depends on firm actions but has no further influence on the choice behavior of consumers.

Concerning the costs of attention I assume the following properties to hold for $f \in [0, \infty)$

$$C(0) = 0, \ \ C(f > 0) \in (0, \infty), \ \ C'(0) \in [0, \infty), \ \ C'(f > 0) \in (0, \infty), \ \ C''(f) \in [0, \infty) \tag{3.7}$$

An example which provides useful later is given by

$$C(f) = \theta f^{\eta} \qquad \theta > 0 \quad \eta \geq 1 \tag{3.8}$$

As the elasticity of the cost function will play a major role for the comparative statics of the model it is appropriate to discuss the cost function in greater detail which is accomplished in the remainder of this section. In case of sponsored search advertising firms can purchase certain keywords (called adwords) by offering a cost-per-click (CPC) rate in an ongoing online auction. For a purchased keyword the rank of the link within the frame for sponsored ads depends on the relative bid and on the PageRank measure of the page[13]. Thus for a given set of keywords and identifying attention probabilities with the on-screen ranking we might expect constant unit costs to be reasonable. Imagine a situation where all initial bids are the same for a fixed set of keywords. Assuming that only relative bids determine the ranking the on-screen position then is random and the probability to be among the first R ranks is R/n for every competitor. Then if all competitors double their bids my firm can get the same chance as before (R/n) only if I also double my bid which simply implies doubling advertising expenditure. However, as Williamson and Rusmevichientong point out, sponsored search advertising is highly complex as it is a multidimensional issue if the set of keywords is not fixed and novel keywords are associated with an unknown number of click-through rates (Williamson and Rusmevichientong (2006), p. 260). Nonlinear costs of attention may arise e.g. if retrieving new click-generating keywords gets more difficult the more keywords are already employed. Nonlinearity is more generally also supported by the finding of Nothdurft that it gets increasingly difficult to generate a higher salience (pop-up effect) of some visual stimulus at higher levels of salience (Nothdurft (2000), p. 1195): combining the salience

[13] In case of Google the CPC corresponds to the next highest bid plus one cent.

of two features (e.g. color and movement of a visual object) does not lead to the sum of the saliences of the two features for the same background. Finally, in case of adword competition how narrow the set of possible keywords is might be a market-specific feature which then implies different elasticities of the cost functions for different markets. The market for a specialised set of screwdrivers might encompass less critical keywords than the market of holiday destinations.

3.2.3 The symmetric price-attention game

Let $S_y \equiv [c, y^{max}]$, $S_f \equiv [0, \infty)$ and $S \equiv [c, y^{max}] \times [0, \infty)$. Further, I assume (3.5) to be symmetric, i.e. (see chapter 5.2.1)

$$\Pi^j \left((y_1, f_1), ..., (y_j, f_j), ..., (y_n, f_n)\right) = \Pi^{\sigma(j)} \left((y_{\sigma(1)}, f_{\sigma(1)}), ..., (y_{\sigma(j)}, f_{\sigma(j)}), ..., (y_{\sigma(n)}, f_{\sigma(n)})\right)$$

where σ is a permutation of $\{1, ..., n\}$ All active firms simultaneously and non-cooperatively choose their strategy, the pair $(y_j, f_j) \in S$ in order to maximize (3.5) and take (y_{-j}, f_{-j}) as given. Hence (n, S^n, Π) is a static symmetric game. In the main part of this chapter I restrict myself to the case of symmetric equilibria.[14] To establish a symmetric equilibrium I apply the symmetric opponents form approach (SOFA, see chapter 5.2.1). In the next section I derive the symmetric opponent form of (3.5). It is convenient to impose all further assumptions relevant for the analysis of the symmetric equilibrium directly on the symmetric opponent form.

3.2.3.1 The symmetric opponent form

To derive the symmetric opponent form of the profit function of (3.5) I take firm j as the representative firm and set $y_j = y$ and $y_g = \bar{y}$ as well as $f_j = f$ and $f_g = \bar{f}$ for any $g \neq j$. Then $V^j(A) = V^j(A')$ if $A, A' \in B_j$ and I define $V(y, \bar{y}, z) \equiv V^j(Y_A : A \in B_j)$ (remember

[14]Appendix A explores the possibility of asymmetric equilibria in the symmetric game.

that the number of arguments in V^j is z) and[15]

$$p\left(f, \bar{f}, n, R\right) \equiv \begin{cases} 1 & n \leq R \\ p_j(f, f_{-j})|_{f_g=\bar{f}\;\forall\, g\neq j} & n > R \end{cases}$$

Lemma 3.2. *If $n > R$ the symmetric opponent form of* (3.5) *is*

$$\Pi(y, f) = p(f, \bar{f}, n, R) V\left(y, \bar{y}, R\right) \Delta - F - C(f) \tag{3.9}$$

If $R \geq n$ the symmetric opponent form of (3.5) *is*

$$\Pi(y, f) = V\left(y, \bar{y}, n\right) \Delta - F - C(f) \tag{3.10}$$

Proof: Appendix B (3.7.4)

Note that $p(f, \bar{f}, n, R)\Delta$ is the fraction of consumers which perceive firm j. That is, $p(f, \bar{f}, n, R)\Delta$ is the number of realised attention sets $\tilde{A}$ that contain j.

Because $C(f)$ is an injective function of f it is possible to rewrite (3.9) directly in terms of attention costs rather than in terms of attention effort f. For $f = \rho(\omega)$ with $C\left(\rho(\omega)\right) = \omega$ and $\bar{f} = \rho(\bar{\omega})$ with $C\left(\rho(\bar{\omega})\right) = \bar{\omega}$ we get

$$\Pi(y, \omega) = p\left(\rho(\omega), \rho(\bar{\omega}), n, R\right) V(y, y, R)\Delta - F - \omega$$

This reformulation makes sense as firms rather observe attention costs than attention effort of their opponents. However, it is convenient to solve the model in terms of effort f rather than in terms of attention cost and I continue to use this specification.

It is possible to establish a simple relationship between the probability p that firm j is perceived by a consumer and the probability, $\bar{p}$, that one of its opponents is perceived. Let p_g be the probability that $g \neq j$ is perceived by a consumer. Then $f_g = \bar{f}$ for all

[15] For $R < n$ the function $p(f, \bar{f}, n, R)$ generally depends on R because there are $|\mathcal{A}| = \binom{n}{R}$ possible attention sets and the probability assignment as defined by (3.2) depends on how many attention sets there are.

$g \neq j$ implies that $p_g = \bar{p}$ for all $g \neq j$.[16] Proposition 3.1 shows that

$$\bar{p} = \frac{R-p}{n-1} \qquad R < n \tag{3.3'}$$

3.2.3.2 Assumptions on the symmetric opponent form

In this section I impose and discuss the main assumption on the value function $V(y, \bar{y}, z)$ and the probability function $p(f, \bar{f}, n, R)$. I take these assumptions to be satisfied in the main part of this chapter.

Assumption 3.3. *The function $V(y, \bar{y}, z)$ is twice continuously differentiable in y, $\bar{y}$ for $y, \bar{y} \in S_y$ and decreasing in z. $V_2(y, \bar{y}, z) > 0$.*

The assumption $V_2(y, \bar{y}, z) > 0$ means that higher prices of the opponents ceteris paribus increase the revenue that the firm can extract from its attention sets. In case of (3.4) this means that higher prices of the opponents ceteris paribus increases demand (as the opponents loose some consumers to the firm) and hence also revenue for the firm. The assumption that $V(y, \bar{y}, z)$ decreases in z means that if consumers perceive more alternatives then ceteris paribus the firm earns less value from the consumer. In case of (3.4) a standard intuition for this is that consumers allocate their budget over the set of all firms they perceive. If they perceive more firms then, for a fixed budget v, they divide their budget over more firms which leaves less budget for a single firm.

As an example for a function that satisfies assumption 3.3 suppose consumers have CES-preferences over all z commodities they perceive and $c > 0$. Then it is straightforward to show that

$$V(y, \bar{y}, z) = (y - c) \underbrace{\frac{vy^{-\sigma}}{y^{1-\sigma} + (z-1)\bar{y}^{1-\sigma}}}_{x(y,\bar{y},z)} \tag{3.11}$$

where $x(y, \bar{y}, z)$ denotes demand of a consumer for commodity j who perceives j and also $z - 1$ other firms. The parameter $\sigma > 1$ denotes the elasticity of substitution[17] and $v > 0$

[16] Moreover, if we know $p(f, \bar{f}, n)$ then we can determine $P_A(\mathcal{F})$ for all $A \in B_j$ as well as $P_{A'}(\mathcal{F})$ where $A' \notin B_j$ (see the remark in section 3.7.4 of the appendix). This means that, if we restrict ourselves to the symmetric opponent form, whenever we determine $p(f, \bar{f}, n)$ then the probability set function P is also determined which means that every possible attention set $A \in \mathcal{A}$ has a well defined probability of being realised.

[17] $V_2(y, \bar{y}, z) > 0$ follows because $\sigma > 1$.

is a consumer's budget.

Assumption 3.4. *Let $p \in [0,1]$ denote the probability of firm j of being in the attention set of a consumer.*

a)

$$p = p\left(f, \bar{f}, n, R\right) \tag{3.12}$$

If $n > R$, $f, \bar{f} > 0$ and $p < 1$ then the following properties of the function $p\left(f, \bar{f}, n, R\right)$ are postulated:

b) $p\left(\lambda f, \lambda \bar{f}, n, R\right) = p\left(f, \bar{f}, n, R\right)$ for $\lambda > 0$.

c) $p\left(f, \bar{f}, n, R\right)$ is twice continuously differentiable in f, $\bar{f}$ and

$$p_1(f, \bar{f}, n, R) > 0 \quad p_{11}(f, \bar{f}, n, R) < 0 \quad p_2(f, \bar{f}, n, R) < 0$$

Moreover, $p(f, \bar{f}, n, R)$ is strictly increasing in R and strictly decreasing in n.

P1) of the introduction (chapter 1.2) implies that how successful a firm is in capturing a consumer's attention depends on its effort compared to total effort and is a relative matter. Hence the effort level f should affect the probability to get attention, p, in a relative way as is stated by b).[18] Because of assumption c) $p(f, \bar{f}, n, R)$ is an increasing strongly concave function of f. Note that $p_1 > 0$ directly follows from lemma 3.1 if we assume the function $p(f, \bar{f}, n, R)$ to be differentiable in f. The assumption that $p_2 < 0$ means that attention efforts impose a negative externality of senders on each other.[19] The assumption that $p(f, \bar{f}, n, R)$ is strictly increasing in R means that the probability to be perceived ceteris paribus *increases* with the size of the attention set which is intuitively

[18]Falkinger was the first to embed the importance of relative signal strength for perception into an economic model (Falkinger (2007) and Falkinger (2008)).

[19]Generally, a positive effect of aggregate attention effort, $\Sigma = \sum_{j=1}^{n} f_j$ on a market could be imagined if Δ were not exogenously fixed but depends on Σ. Think of a market where by some reason fierce attention competition between the firms for those consumers that are already aware of the market leads to a high Σ. But this may imply that information about the market leads to a strong general coverage in the media which may attract new consumers to that market *without* the firms explicitly wooing them. If there is an aggregate effect of attention competition in the sense that louder markets attract more consumers (have a higher Δ) then attention driven spillover effects between markets may exist. This is the subject of my mimeo "Attention markets".

clear: if people perceive more (less) alternatives then, ceteris paribus, the probability of a single firm to be perceived should increase (decrease). Similarly, the assumption that $p(f, \bar{f}, n, R)$ decreases in n means that if there are more (less) active firms then, ceteris paribus, the chance of a single firm to be perceived should decrease (increase).

If $n > R$ and all active firms choose the same effort level f then the probability of being perceived by a consumer is the same for all firms:

Lemma 3.3. *If $n > R$ and $f = \bar{f}$ then $p = \bar{p} = R/n$.*

Proof: $f = \bar{f}$ implies $p = \bar{p}$. The result then follows immediately from (3.3').

■

3.2.3.3 An example: The Attention Contest Function

In this section I present an important example of a function $p(f, \bar{f}, n, R)$ that satisfies assumption 3.4. Besides having a nice intuition this function also has some analytically convenient properties. Let

$$p(f, \bar{f}, n, R) = 1 - \prod_{i=1}^{R} \left(1 - \frac{f}{f + (n-i)\bar{f}}\right) \tag{3.13}$$

Throughout my thesis I refer to (3.13) as the (symmetric) Attention Contest Function (ACF).

Proposition 3.2. *The symmetric ACF satisfies all properties from assumption 3.4.*

Proof: Appendix B (3.7.5)

Note that we cannot partially differentiate the ACF as stated in (3.13) with respect to R because the function is not continuous in R.[20]

The remainder of this section provides an intuitive derivation of the ACF. The following derivation is from the perspective of firm j. Let

$$F_n = \frac{f_j}{\sum_{k=1}^{n} f_k} \tag{3.14}$$

[20] It is formally possible to extend (3.13) to a continuous function in R by using a Gamma function expansion. See the appendix for details.

measure the relative effort of firm j. Now assume that getting a consumer's attention can be described by a stochastic process similar to making random draws without repetition in a simple urn model. Hence the consumer will draw R times from this urn which corresponds to selecting R different alternatives. Firm j is in the attention set of a consumer if it gets either the first, the second ... or the R-th draw. F_n corresponds to the probability of getting the first of the R draws. This probability depends (positively) on the mass of firm j's ball (f_j) versus the aggregate mass of all balls in the urn ($\sum_{k=1}^{n} f_k$). If $f_k = \bar{f}$ and $f_j = f$ then (3.14) becomes[21]

$$F_n = \frac{f}{f + (n-1)\bar{f}}$$

and $1 - F_n$ corresponds to the probability of not getting the first draw. Similarly,

$$(1 - F_n)(1 - F_{n-1}) = \left(1 - \frac{f}{f + (n-1)\bar{f}}\right)\left(1 - \frac{f}{f + (n-2)\bar{f}}\right)$$

is the probability of not getting the first and second draw (which obviously never is larger than $1 - F_n$) and generally $\prod_{i=1}^{R} (1 - F_{1+n-i})$ is the probability of not getting any of the R draws. Consequently, the probability of being in one of the R draws is given by (3.13).

3.2.3.4 Best response of the representative firm

In this section I discuss and illustrate the best-response function of the representative firm. Suppose that $n > R$. Assuming an interior solution the two first-order conditions of (3.9) are

$$\begin{aligned} &V_1(y, \bar{y}, R) = 0 \\ &p_1\left(f, \bar{f}, n, R\right) \Delta V(y, \bar{y}, R) = C'(f) \end{aligned} \tag{3.15}$$

From (3.15) we see that $y = y(\bar{y}, R)$ and $f = f(\bar{f}, \bar{y}, n, R, \Delta)$. This means that the strategic choice of price of our firm is independent of attention efforts but it depends on the perceived market size (R) and not on the total amount of information that a consumer receives (which is $|I| = n$). This is the same result as in the last chapter[22] (see 2.3.1). Also note that f depends on n because the marginal probability of perception depends

[21] In Appendix A (3.6.1) I discuss the ACF without imposing $f_k = \bar{f}$. This is technically necessary when dealing with the possibility of asymmetric equilibria.

[22] Note that in this chapter we did not require any type of approximation for this result.

on n and f depends on R because *both* the marginal probability and the value function V depend on R. This will have an important consequence for the comparative statics of the symmetric equilibrium later. Because the marginal probability of perception decreases in f (assumption 3.4) and $V_2(y, \bar{y}, R) > 0$ (assumption 3.3) we immediately get $f'(\Delta) > 0$ and also $f'(\bar{y}) > 0$.[23] More consumers (a larger audience) means more potential budgets which increases the marginal value of attention and leads to a higher effort level. Similarly, if $\bar{y}$ increases this means that our firm can earn more value from every attention set that it is contained in which increases the marginal value of attention and hence also effort level f. In the CES-example (3.11) we get $y'(\bar{y}) > 0$ and also $y'(R) < 0$ (see section 3.7.6, Part I, for the calculations). The last result means that, because consumers are less attentive, our firm sets a higher price and originates from the fact that if a consumer has a fixed budget but perceives less commodities then he spends his funds over fewer firms. This means that, ceteris paribus, the demand for the commodity of our firm increases and the optimal action of the firm then is to exploit this by setting a higher price.[24] How does f depend on $\bar{f}$? Formally, the answer to this question depends on how $\bar{f}$ affects the marginal chance of perception $p_1(f, \bar{f}, n, R)$:

$$f'(\bar{f}) = \frac{p_{12}(f, \bar{f}, n, R)\overbrace{V(y, \bar{y}, R)\Delta}^{>0}}{\underbrace{C''(f) - p_{11}(f, \bar{f}, n, R)V(y, \bar{y}, R)\Delta}_{>0}}$$

Intuitively, we could imagine $\bar{f}$ to have a complementary effect on f by a defensive-type of argument: if all opponents increase their effort level then the representative firm must also increase its effort level in order to remain salient. But we could also expect that more "noise" means that it gets more difficult for our firm to influence its chance of perception. In such a case we would expect that $p_1(f, \bar{f}, n, R)$ decreases in $\bar{f}$. It turns out that the ACF incorporates both features which implies a non-monotonic relationship between $\bar{f}$ and f. Figure 3.4 depicts f as a function of $\bar{f}$ and n in case of the ACF with $R = 2$ and

[23] $f'(\bar{y}) = \frac{p_1 V_2 \Delta}{C'' - p_{11} V \Delta} > 0$

[24] Note that such an effect cannot exist in Falkinger's model of limited attention (which also uses the CES demand function) as firms have no mass and the price does not depend on the measure of the perceived market size (Falkinger (2008)).

cost function[25] (3.8). What these pictures generally tell us is that with the ACF if there

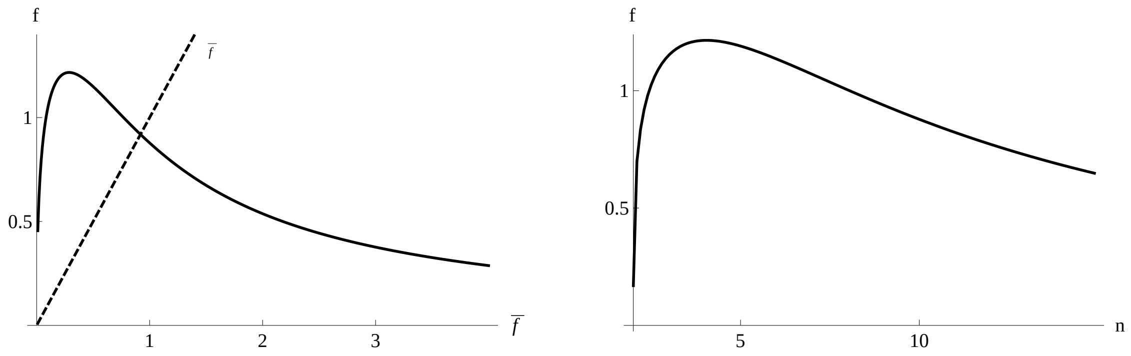

Figure 3.4: ACF ($n = 10$ left, $\bar{f} = 1$ right)

is little aggregate attentional activity of the opponents ("low noise"), because $\bar{f}$ is small or n is small (close to R), then f increases in the noise whereas if the level of noise is already high then the opposite occurs.

3.3 Existence of a symmetric equilibrium

In this section I derive the conditions which assert the existence of exactly one symmetric equilibrium in the price-attention game as described in section 3.2.3. I start with the case where n is given exogenously and then proceed to the case where n is endogenously determined by the free-entry condition of the two-stage game. I show that the CES-ACF example has exactly one symmetric equilibrium.

3.3.1 Symmetric equilibrium for exogenous n

The fundamentals of the symmetric game are the parameter set $\{n, R, \Delta\}$, the cost function $C(f)$, the value function $V(y, \bar{y}, z)$ with $z = min\{R, n\}$ as defined by assumption 3.3 and the probability function $p(f, \bar{f}, n, R)$ as defined by assumption 3.4. In a symmetric equilibrium all n firms choose the same strategy (y, f). To find a symmetric equilibrium we can rely on the SOFA (see chapter 5.2.1).

[25]I impose $V(y, \bar{y}, 2)\Delta = 10$, $\theta = 1$ and $\eta = 2$. The pictures do not change qualitatively if other numerical values are used (or if $R > 2$).

Suppose that $n > R$. Assuming an interior solution, the first-order conditions of the representative firm's optimisation problem are given by (3.15). Then, a solution (y, f) to

$$\begin{gathered} V_1(y, y, R) = 0 \\ p_1(f, f, n, R)\, \Delta V(y, y, R) = C'(f) \end{gathered} \tag{3.16}$$

corresponds to an interior symmetric equilibrium of the game.

If $R \geq n$ then it is easy to see from (3.10) and (3.7) that a symmetric equilibrium (y, f) of the game with $y \in (c, y^{max})$ must have $f = 0$ and satisfies $V_1(y, y, n) = 0$.

To summarise: for given $R, n > 1$ and $\Delta > 0$ a symmetric equilibrium (y, f) of the price-attention game satisfies

$$\begin{gathered} V_1(y, y, z) = 0 \\ f : \begin{cases} p_1(f, f, n, R)\Delta V(y, y, R) - C'(f) = 0 & R < n \\ f = 0 & R \geq n \end{cases} \\ z = \min\{R, n\} \end{gathered} \tag{3.17}$$

Assumption 3.5. *a) The following boundary conditions are satisfied:*

i) For any $z > 1$ we have $V_1(c, c, z) > 0$ and $V_1(y^{max}, y^{max}, z) < 0$.

ii) If $n > R$ we have $\lim_{f \to 0} p_1(f, f, n, R) = \infty$ and $\lim_{f \to \infty} p_1(f, f, n, R) = 0$

b) For any $z > 1$ and $y \in (c, y^{max})$ we have $V(y, y, z) \in (0, \infty)$, $V_{11}(y, y, z) < 0$ and[26]

$$V_1(y, y, z) = 0 \quad \Rightarrow \quad -\frac{\partial}{\partial y}\left(V_1(y, y, z)\right) > 0 \tag{3.18}$$

c) For any $R > 1$ and any n with $n > R$ and $f \in (0, \infty)$

$$-\frac{\partial}{\partial f}\left(p_1(f, f, n, R)\right) > 0 \tag{3.19}$$

The boundary conditions exclude the possibility of symmetric boundary equilibria which simplifies the comparative-static analysis later. The assumptions of monotony (3.18) and (3.19) exclude the possibility of multiple symmetric equilibria.

[26] Note that $\frac{\partial}{\partial y}(V_1(y, y, z)) = V_{11}(y, y, z) + V_{12}(y, y, z)$, i.e. $\frac{\partial}{\partial y}(V_1(y, y, z))$ is the *total derivative* of $V_1(y, y, z)$ with respect to y. Similarly, $\frac{\partial}{\partial f}(p_1(f, f, n, R))$ is that total derivative of $p_1(f, f, n, R)$ with respect to f.

Proposition 3.3. *If assumption 3.5 is satisfied then for any $R, n > 1$ there exists a unique vector (y, f) that solves* (3.17)*. Hence there exists only one symmetric equilibrium. In the equilibrium we have $y \in (c, y^{max})$. If $R \geq n$ we have $f = 0$ and $y = y(n)$. If $n > R$ then $y = y(R)$ and $f = f(n, R, \Delta) \in (0, \infty)$.*

Proof: Appendix B (3.7.7)

Hence if $R \geq n$ then all active firms choose the minimal effort level $f = 0$. Because in this case the consumer perceives all alternatives he is informed of there is no benefit from choosing a higher level of f.

A nice property of the ACF is that this functional form for $p(f, \bar{f}, n, R)$ satisfies the boundary conditions in assumption 3.5 as well as condition (3.19).

Lemma 3.4. *Suppose $n > R$ and the function $p(f, \bar{f}, n, R)$ is given by the ACF* (3.13)*. Then for $f > 0$*

$$p_1(f, f, n, R) = \frac{n-R}{nf} \sum_{i=1}^{R} \frac{1}{1+n-i} \tag{3.20}$$

The ACF satisfies assumption 3.5. A good approximation to (3.20) *if n is large relative to R is given by*

$$p_1(f, f, n, R) \cong \frac{n-R}{n^2 f} R \tag{3.21}$$

Proof: Appendix B (3.7.8)

Almost all results in this paper are proved by using (3.20) and not its approximation. In any case the results hold also if (3.21) were used instead and the proofs usually are less technical. In applications we can use (3.21) rather than (3.20) because (3.21) is differentiable also in R and generally simpler to work with. Because the ACF satisfies assumption 3.5 then whenever we combine the ACF with a function $V(y, y, R)$ that satisfies assumption 3.5 we always end up with a single symmetric equilibrium (for $n > R$).

Corollary 3.1. *If for $n > R$ the function $p(f, \bar{f}, n, R)$ is given by the ACF and $V(y, y, R)$ satisfies assumption 3.5 then there exists exactly one symmetric equilibrium $(y, f) \in$*

$Int(S)$. f is approximately determined by

$$\frac{n-R}{n^2 f} R \Delta V(y,y,R) = C'(f) \tag{3.22}$$

Proof: Follows from proposition 3.3, lemma 3.4 and (3.21).

■

3.3.2 Free-entry equilibrium: existence and uniqueness

In the symmetric equilibrium the level of profits is determined by

$$\Pi(y,f,n) = \begin{cases} \Delta V(y,y,n) - F - C(f) & n \leq R \\ \frac{R}{n}\Delta V(y,y,R) - F - C(f) & n > R \end{cases} \tag{3.23}$$

for any given $R > 1$. The entry game then is a two-stage game where firms first decide whether or not to enter. A firm that chooses to enter pays the fixed costs $F > 0$. All firms that chose to enter then play the symmetric price-attention game from the last section. In a pure subgame perfect equilibrium (SPE) given any decision in the first stage a Nash equilibrium follows in the second stage. In a free-entry equilibrium each active firm makes non-negative profits ($\Pi(n) \geq 0$) and further entry would result in negative profits ($\Pi(n+1) < 0$). Concerning the Nash-equilibrium of the second stage we already know that for any given $n, R > 1$ there exists a single symmetric equilibrium. Ignoring the integer-value problem of n the free-entry equilibrium (y, f, n) then is described by

$$V_1\left(y, y, min\left\{R, n\right\}\right) = 0 \tag{3.24}$$

$$f: \begin{cases} p_1(f,f,n,R)\Delta V(y,y,R) - C'(f) = 0 & R < n \\ f = 0 & R \geq n \end{cases} \tag{3.25}$$

$$\Pi(f,y,n) = 0 \tag{3.26}$$

With symmetry we can only hope to determine n, the number of active firms in a SPE and not which firm enters and which firm stays out. I call a SPE unique if there only is one vector (y, f, n) that solves (3.24) - (3.26). The fundamentals of the symmetric game with free entry are the parameter set $\{R, \Delta\}$, the cost function $C(f)$, the value function

$V(y, \bar{y}, z)$ with $z = min\{R, n\}$ as defined by assumption 3.3 and the probability function $p(f, \bar{f}, n, R)$ as defined by 3.4.

Assumption 3.6. *The following properties are imposed:*

a) The function $V(y, y, z)$ is twice continuously differentiable in z and $V_3 < 0$ as well as $V_{13} < 0$.

b) The function $p_1(f, f, n, R)$ is continuously differentiable in n

c) For $f > 0$ we have $\lim_{n \to R^+} \; p_1(f, f, n, R) = 0$

The assumption $V_3(y, y, z) < 0$ is just the differentiable version of assumption 3.3 and the last assumption means that the marginal value of an attention set decreases if more alternatives are being considered. Let $\hat{y}$ be implicitly defined by $V_1(\hat{y}, \hat{y}, 2) = 0$.

Proposition 3.4. *Let $2 \leq R < \infty$. Suppose that assumptions 3.5 and 3.6 are satisfied and*

$$\Delta V(\hat{y}, \hat{y}, 2)/F \geq 1 \tag{3.27}$$

If the condition

$$\frac{\frac{\partial}{\partial f}\left(p_1(f, f, n, R)\right)}{p_1(f, f, n, R)} < p_{13}(f, f, n, R)\frac{n^2}{R} \tag{3.28}$$

holds then the free-entry game has a single symmetric SPE (y, f, n) with $2 \leq n < \infty$

Proof: Appendix B (3.7.9)

To understand intuitively why condition (3.28) excludes the possibility of multiple symmetric equilibria note that (3.28) is satisfied if $p_{13} > 0$. Suppose that $p_{13} > 0$ holds for any $n > n'$ and assume that (y', f', n') is an equilibrium of the free-entry game with $n' > R$. From (3.16) it is not difficult to see that $p_{13} > 0$ implies that $f'(n) > 0$ for $n \geq n'$. That is, equilibrium attention effort increases if we *exogenously* increase n. From (3.23) we see that an exogenous increase of n has two effects on the value of $\Pi(y(n), f(n), n)$: equilibrium profits decrease in n because R/n decreases in n. This is the direct effect of n on the profit level and originates from the fact that more senders means that *equilibrium* chances of being perceived decrease for a single firm which means less (expected) revenue. But there also is an indirect effect on the level of profits because $C(f)$ depends on n in the equilibrium. $f'(n) > 0$ implies that attention costs increase in n which means that profits decrease in n also over this channel. Thus we see that $p_{13} > 0$ unambiguously implies profits to decrease in n for $n > n'$. But as we have n' in the free-entry equilibrium this means that profits must be negative for any $n > n'$. Consequently, there cannot exist a free-entry equilibrium with $n > n'$. This is illustrated in the left picture of figure 3.5. If

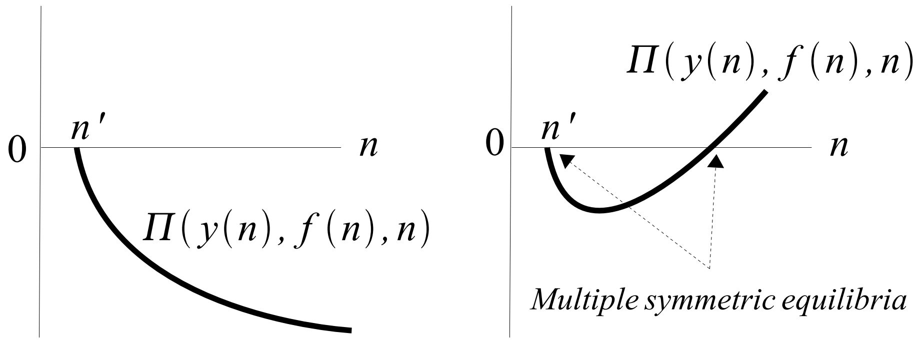

Figure 3.5: The possibility of multiple symmetric equilibria under limited attention

however we have $p_{13} < 0$ then the indirect effect of a change of n on profits is positive as a higher n means lower f and thus lower costs $C(f)$. If this indirect effect is very strong then profits might also increase in n. In this case we might have multiple symmetric equilibria as is illustrated in the right picture of figure 3.5.

A very nice property of the ACF is that condition (3.28) holds.

Corollary 3.2. *Let* $2 \leq R < \infty$. *The ACF (see* (3.20)*) satisfies assumption 3.6. If* $V(y, y, z)$ *satisfies assumptions 3.5 and 3.6, the function* $p(f, f, n, R)$ *is given by the ACF and condition* (3.27) *holds then the free-entry game has a single symmetric SPE* (y, f, n) *with* $2 \leq n < \infty$

Proof: The ACF satisfies assumption 3.5 because of lemma 3.4. It is not hard to see that the ACF also satisfies b) and c) of assumption 3.6. Finally, the ACF also satisfies condition (3.28) (see lemma 3.7 in the appendix (3.7.1)). The claim then follows from proposition 3.4.

■

3.3.3 The ACF-CES example

Suppose that $n > R$ is given exogenously. If $p(f, \bar{f}, n, R)$ is given by the ACF and $V(y, \bar{y}, z)$ by the CES example (3.11) then exactly one symmetric equilibrium (y, f) exists and is approximately[27] determined by (see 3.7.6, Part II, for the calculations)

$$y = c + \frac{cR}{(R-1)(\sigma-1)} \tag{3.29}$$

$$\frac{(n-R)Rv\Delta}{fn^2(1+(R-1)\sigma)} = C'(f) \tag{3.30}$$

Moreover, we have

$$V(y, y, z) = \frac{y-c}{y}\frac{v}{z} \tag{3.31}$$

Hence $V_3 < 0$ and $V_{13} < 0$ hold. Suppose that $V(\hat{y}, \hat{y}, 2) = \frac{v\Delta}{1+\sigma} \geq F$. Then by corollary 3.2 we may conclude that in the ACF-CES example there exists exactly one symmetric SPE with $n \geq 2$.

3.4 Comparative statics

In this section I discuss the comparative-static results of the symmetric price-attention equilibrium. I begin with an exogenous number of firms and then proceed to the case where n is determined endogenously in a free-entry equilibrium. I always take assumptions 3.5 and 3.6 to be satisfied. I extend the cost function from (3.7) by a parameter α, i.e. $C = C(f, \alpha)$ with the property that $C_2(f, \alpha) > 0$ and $C_{12}(f, \alpha) > 0$. For notational simplicity I set $C' \equiv C_1(f, \alpha)$ and $C'' \equiv C_{11}(f, \alpha)$.

3.4.1 Comparative statics for an exogenous number of firms

In this section I derive the comparative statics of the symmetric price-attention game where n is given exogenously. I only discuss the case where $n > R$ is given exogenously. The equilibrium conditions are given by (3.16). The game has the following set of exogenous parameters: $\{R, n, \Delta, F, \alpha\}$.

[27] Note that (3.29) is exact and (3.30) follows from using (3.21).

Proposition 3.5. *The comparative statics of* (3.16) *are given by* $y = y(\underset{-}{R})$ *and* $f = f(\underset{?}{R}, \underset{?}{n}, \underset{+}{\Delta}, \underset{-}{\alpha})$.

Proof: Appendix B (3.7.10)

As in chapter 2 the only parameter that matters for equilibrium prices is the size of the attention set. $y'(R) < 0$ results because $V_{13}(y, y, R) < 0$. This assumption means that the equilibrium marginal value that can be earned from an attention set decreases if people compare more alternatives. This is a very intuitive assumption and many examples (such as the CES example and the Salop-Shapiro-Grossman example from chapter 4.1) satisfy this property. In the CES case this follows because consumers spend their fixed budget v over more alternatives if they perceive a larger set which means that less budget is available for the single firm. To offset this loss of demand the best-response is to decrease the price which explains why equilibrium prices are lower if the perceived market is larger.

However, the fact that y is determined only by the perceived market size and does not depend on other parameters implies that the aggregate level of attention competition nf (e.g. advertising intensity) observed may be a bad predictor of market power. Suppose we have two almost identical economies $\mathcal{E}_1 = \{R, n, \Delta, F, \alpha_1\}$ and $\mathcal{E}_2 = \{R, n, \Delta, F, \alpha_2\}$ with $\alpha_1 > \alpha_2$. Then $nf_1 < nf_2$. The classical theory of informative advertising holds that higher aggregate advertising levels should deteriorate market power of the firms which in equilibrium manifests itself in lower prices (see chapter 2). The intuition behind this theory is that *limited information* of consumers because of scarce information leaves the firm with market power (see Goeree (2008)). In such a case means to increase advertising intensity would be beneficial for consumers as competition is reinforced and prices are decreased. We see that under limited attention this conjecture does not hold as consumers have limited information because of their limited attention capacities and *not* because of scarce information.

Search models usually predict lower equilibrium prices if search costs are reduced and electronic marketplaces or the internet with its powerful search engines are generally thought of dramatically lowering the search costs (e.g. Bakos (1997)). In such models this usually means that people become aware of more (or better) alternatives. However, in many cases the anticipated price-reduction was a lot smaller as suggested by these models (Ellison and Ellison (2005), p.149) or on the contrary online prices even were found to be higher than offline prices (e.g. Lee (1998)). The model of this paper provides an explanation why such gaps between theory and empirics exist. Search models do not distinguish the information set from the attention set. If we agree to $V_{13} < 0$ then larger attention sets will always imply lower prices in such models. If however the attention set is a subset of the information set with fixed size then larger information sets have no (or at least less) impact on equilibrium prices. The massive evidence on top-rank clicking-behaviour and also on the fact that people often do not use more than one search query (e.g. Jansen et al. (2000), p. 212) means that people concentrate their decision on just a few alternatives which suggests that not search cost theories but an attention theory adequately models the contemporaneous and future information problems.

As is shown in the appendix (see 3.7.10) we have $sign(f'(n)) = sign\,(p_{13})$ and $sign\,(f'(R)) = sign\left(p_1 V_2 V_{13} - \frac{\partial}{\partial y}\left(V(y,y,R)\right)(p_{14}V + p_1 V_3)\right)$. These expressions cannot be signed without further assumptions. In case of a change of R there are two effects at work[28]. First, a decrease of R increases the equilibrium value of an attention set because i) an attention set must be shared with fewer competitors and ii) equilibrium prices are higher[29]. This set value effect increases the marginal revenue of attention and increases attention levels f. Second, the marginal probability of attention also depends on R. Depending on the sign of p_{14} the effect of R on p_1 may either reinforce or offset the set value effect. A sufficient condition for $f'(R) < 0$ is that $\frac{p_{14}R}{p_1} < \frac{-V_3 R}{V}$ holds in equilibrium. If $V(y,y,R) = \tilde{V}(y)/R$ with $\tilde{V}(y)' > 0$ then[30] the RHS of this inequality is one. As for the ACF $p_1(f,f,n,R)$ is a strongly concave function of R which goes through the origin (see (3.21) in case of the approximation and lemma 3.6 in the appendix for the general case) we may conclude that $f'(R) < 0$ in such a case because $\frac{p_{14}R}{p_1} < 1$. Hence less attention implies higher prices and higher effort levels (stronger attention competition) in this example.

Suppose now that n increases exogenously. We see that the sign of $f'(n)$ depends entirely on how the marginal probability p_1 depends on n. This is different from the case of an exogenous change of R as a change of n has no set value effect. In case of the ACF

[28] Recall the discussion of strategic behavior in section 3.2.3.4

[29] Formally: $\frac{dV}{dy} = V_3 + V_2 y'(R) < 0$.

[30] This is a property that many demand functions share, e.g. demand derived from the circular ideal variety model (see lemma 4.1 in chapter 4.1). Also the CES example has this property: $V(y,y,R) = \frac{v(y-c)}{Ry} = \frac{(y-c)}{y}\frac{v}{R}$.

it can be shown (see 3.7 in the appendix (3.7.1)) that $p_1(f, f, n, R)$ is a hump-shaped function of n. Consequently, $f(n)$ takes on a hump-shaped form in this example.

Summarising, we see that limited attention implies equilibrium prices to be independent of the size of the information set (n) which is different from models of informative advertising. In equilibrium, prices reflect the scarcity of attention in the sense that smaller attention sets mean higher prices. On the other side limited attention implies that a competition for attention emerges ($f > 0$ if $n > R$). If consumers are less attentive this means that the firms can earn higher values. But this makes attention more valuable for the firms which tends to increase attention levels - and attention costs. In case of the ACF this is the dominant effect which means that less attention always leads to higher attention costs. In this case we have two effects on profits if consumers are more inattentive: i) the set value effect which states that more inattention (lower R) increases profits and ii) the fact that, because getting attention is more valuable, the competition for attention is reinforced and leads to higher attention costs.

In the next section I investigate the implications of these two conflictive effects for the equilibrium number of firms that survive the price-attention competition.

3.4.2 Comparative statics in the free-entry game

In this section I discuss the consequences of limited attention for the equilibrium (y, f, n) of the free-entry game. The equilibrium conditions are given by (3.23) - (3.26). The game has the following set of exogenous parameters: $\{R, \Delta, F, \alpha\}$. In the entire section I assume that (3.27) and (3.28) are satisfied and $R \geq 2$. To understand intuitively how n depends on the these parameters it is sufficient to understand how the profit function depends on the parameters because $sign\,(\Pi'(\chi)) = sign\,(n'(\chi))$ for $\chi \in \{R, \Delta, F, \alpha\}$

3.4.2.1 Comparative statics and conditions for endogenous limited attention

An attention equilibrium (an equilibrium with $n > R$) occurs endogenously in the free-entry game if the system

$$\begin{aligned} &V_1(\tilde{y}, \tilde{y}, \tilde{n}) = 0 \\ &\Delta V(\tilde{y}, \tilde{y}, \tilde{n}) - F = 0 \end{aligned} \tag{3.32}$$

has a solution $(\tilde{y}, \tilde{n})$ with $\tilde{n} > R$. If $\tilde{n} \leq R$ then $(\tilde{y}, 0, \tilde{n})$ corresponds to the equilibrium of the game; a conventional equilibrium occurs. If $\tilde{n} > R$ then there cannot be a conventional symmetric equilibrium. But as exactly one symmetric equilibrium exists this implies that an attention equilibrium must occur. To illustrate the situation graphically note that $\tilde{n} = \tilde{n}(\Delta, F) = R$ for fixed $R \geq 2$ implies that $\frac{d\Delta}{dF} = -\frac{\tilde{n}_2(\Delta,F)}{\tilde{n}_1(\Delta,F)}$. If the Implicit Function Theorem is applied to (3.32) it is straightforward to show that $\frac{d\Delta}{dF} = \frac{1}{V(\tilde{y},\tilde{y},\tilde{n})} > 0$. Figure 3.6 illustrates the situation. In the figure we see that lower infrastructure costs F or a

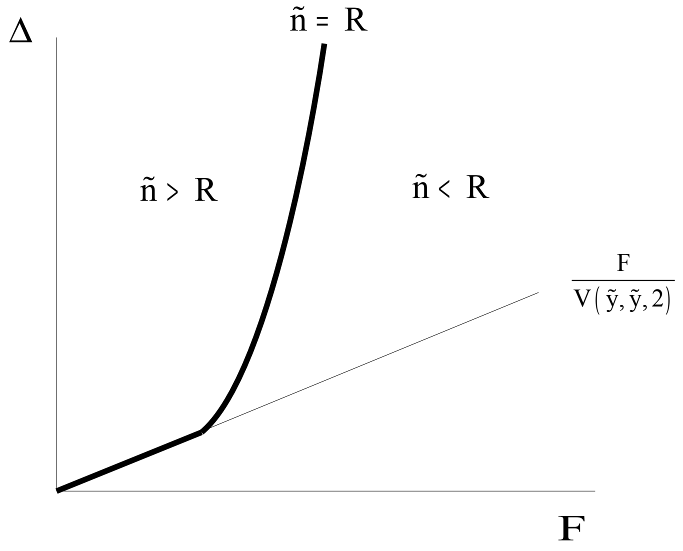

Figure 3.6: Attention equilibrium or conventional equilibrium?

higher market potential Δ rather impose that an attention equilibrium occurs $(\tilde{n} > R)$. Generally, the conditions that imply an attention equilibrium are very similar to the conditions that imply an information-rich economy in Falkinger's model (Falkinger (2008), p. 1604-1605). If we use the CES-example we gain two further parameters: consumer budget $v > 0$ and the elasticity of substitution $\sigma > 1$. Suppose that $V(\tilde{y}, \tilde{y}, 2)\Delta = \frac{v\Delta}{1+\sigma} \geq F$. Then $v\Delta > F$. It is an easy exercise to solve (3.32) in the CES case for $\tilde{n}$:

$$\tilde{n} = \frac{v\Delta}{F\sigma} + \frac{\sigma - 1}{\sigma}$$

Hence we have $\tilde{n}'(v) > 0$ and $\tilde{n}'(\sigma) = \frac{F - v\Delta}{F\sigma^2} < 0$. This means that if the consumers have a higher budget or the commodities are less substitutable then an attention equilibrium is more likely to occur.

3.4.2.2 Comparative statics for limited attention equilibria

In this section I discuss the comparative statics of the symmetric free entry equilibrium if an attention equilibrium occurs endogenously. As in the previous section the comparative statics can be signed unambiguously in many cases if the function $p(f, \bar{f}, n, R)$ is given by the ACF[31]. I show that both f and n increase in the market potential Δ. This means that aggregate attention costs $nC(f)$ also increase in market potential which is consistent with the US-data on the number of consumers and advertising expenditure. Moreover, I show that an increase of the cost parameter α has ambiguous effects on the equilibrium number of firms. A positive relationship between equilibrium profits and the cost parameter α is possible which means that higher costs of attention could lead to a *higher* number of active firms which is an interesting result. Finally, I show that under the ACF the effects of the attention parameter R on f are negative also under free entry, but the effects on n are ambiguous and depend crucially on how much more value can be earned from less attentive consumers compared to how much attention costs increase because of the higher efforts to attract attention.

To simplify notation I set $V'' \equiv \frac{\partial}{\partial y}(V_1(y, y, R))$ and $p'' \equiv \frac{\partial}{\partial f}(p_1(f, f, n, R))$. For the general calculations see the appendix (3.7.11).

Effects of limited attention on prices

Consider two economies $\mathcal{E}_1 = \{R, \Delta_1, F_1, \alpha_1\}$ and $\mathcal{E}_2 = \{R, \Delta_2, F_2, \alpha_2\}$. Assume parameter values such that in the free entry equilibria of these economies we have $n_1 > R > n_2$. Hence in economy one an attention equilibrium occurs whereas in economy two a conventional equilibrium occurs. Then it is straightforward to verify that $y_1 = y_1(R)$ but $y_2 = y_2(\underset{+}{F_2}, \underset{-}{\Delta_2})$. This difference occurs because in an attention equilibrium the competitive entry effect on prices is absent. As in the last section we have $y_1'(R) < 0$.

The sign of $f'(\Delta)$ and $n'(\Delta)$

We know from the last section that the $f(n)$ equilibrium locus can be non-monotonic because of $p_{13}(f, f, n, R) \lesseqgtr 0$ and hence the comparative statics of f in the case of free-entry is not ex ante clear. It can be shown that $sign\,(f'(\Delta)) = sign\left(\frac{p_1(f,f,n,R)}{n} + p_{13}(f, f, n, R)\right)$.

[31] See lemma 3.7 in the appendix (3.7.1) for the properties of the ACF.

That is the sign of $f'(n)$ depends (positively) on the direct effect of Δ on the marginal value of perception but also on how the marginal probability of perception depends on n. In case of the ACF it can be shown that $f'(\Delta) > 0$ (see 3.7). Thus if the market potential increases so does the competition for attention. Further the ACF implies $n'(\Delta) > 0$ because $sign\left(n'(\Delta)\right) = sign\left(-\frac{R}{n}p'' - p_1^2 + \frac{\omega}{n}\right)$ which is unambiguously positive (see 3.7). Hence with the ACF the direct effect on equilibrium profits of a higher market potential always dominates the increase of the costs of attention competition. We know from the data that the number of US-consumers doubled from 1950 - 2005 (see chapter 4.2.3). At the same time advertising expenditure more than doubled (see figure 1.1). This is in line with the comparative static predictions of the model of this chapter as both f and n increase with Δ under the ACF. Hence aggregate advertising expenditure $nC(f)$ (or $nF + nC(f)$) unambiguously increases if Δ increases.[32]

The sign of $f'(F)$ and $n'(F)$

We always have $n'(F) < 0$ unambiguously. In case of the ACF we have that f depends in a hump-shaped way on F because $sign\left(f'(F)\right) = sign\left(-p_{13}\right)$. The different ways in which Δ and F affect f under free-entry originate in the fact that Δ also positively affects the marginal revenue of attention which increases f. In the case of a change of F attention efforts are only affected by the change of n according to the entry or exit decisions and we know from the last section that $f(n)$ is hump-shaped with the ACF.

The sign of $f'(\alpha)$ and $n'(\alpha)$

Let $\varepsilon \equiv \frac{p''f}{p_1} < 0$, $\mu \equiv \frac{C''f}{C'} \geq 0$ and $\beta \equiv \frac{C_{12}f}{C_2} > 0$. Suppose that $p(f, \bar{f}, n, R)$ is given by the ACF. Then it is straightforward to show (use lemma 3.7 b)) that a sufficient condition for $f'(\alpha) < 0$ is given by $\beta + \varepsilon \geq 0$. In case of the ACF we have $\varepsilon = -1$ (see the proof of 3.7 b)). Hence if $\beta \geq 1$ then $f'(\alpha) < 0$. If the cost function is of the type $C(f, \alpha) = \alpha c(f)$ then the sufficient condition for $f'(\alpha) < 0$ reduces to $\frac{c'(f)f}{c(f)} \geq 1$. But $c(0) = 0$, $c'(f) > 0$ and $c''(f) \geq 0$ imply $\frac{c'(f)f}{c(f)} \geq 1$. Hence the ACF together with $C(f, \alpha) = \alpha c(f)$ unambiguously imply that $f'(\alpha) < 0$. Further, it is possible to show that $sign(n'(\alpha)) = sign(\mu - \varepsilon - \beta)$ which is ambiguous. Hence it is possible to have $n'(\alpha) > 0$, which means that increasing

[32]The same holds in the CES example if consumer budget v increases. See section 3.4.2.3.

costs of attention might imply a *higher* number of active firms and is an interesting result. This occurs if the reduction of f, due to the higher marginal costs of attention, lead to lower equilibrium attention costs at the firm level. With the ACF and $C(f,\alpha) = \alpha c(f)$ we get $sign(n'(\alpha)) = sign(\mu + 1 - \beta)$. If $c(f) = f^\eta$ and $\eta \geq 1$ then we have $\mu = \eta - 1$ and $\beta = \eta$. Thus in this case $n'(\alpha) = 0$. Hence in this example α affects only the level of f but has no effect on the equilibrium number of active firms.

<ins>The sign of $f'(R)$ and $n'(R)$</ins>

A decrease in R (consumers are more inattentive) leads to higher value of the attention sets and thus also to a higher marginal return on attention which increases f. But the marginal probability to make it into the attention set also depends on R. It is possible to determine the sign of $f'(R)$ unambiguously if we use the ACF and assume that $V(y, y, R) = \tilde{V}(y)/R$. Then we get

$$sign\left(f'(R)\right) = sign\left(V_{13}RV_2\left(p_{13} + \frac{p_1}{n}\right) - \frac{V''\tilde{V}(y)}{n}\left(p_{14} - \frac{p_1}{R}\right)\right)$$

and hence $f'(R) < 0$ follows unambiguously also with endogenous limited attention (because $p_{14} - \frac{p_1}{R} < 0$ and $p_{13} + \frac{p_1}{n} > 0$, see lemmata 3.6 and 3.7 in the appendix).

To discuss the effect of limited attention on the equilibrium number of firms I set $V(y, y, R) = \tilde{V}(y)/R$ and $C(f) \equiv \theta f^\eta$ with $\theta > 0$ and $\eta \geq 1$. Then with (3.21) the two equilibrium equations (3.25) and (3.26) become

$$\begin{array}{l} \frac{n-R}{n^2\eta}\Delta\tilde{V}(y) = C(f) \\ \frac{\Delta\tilde{V}(y)}{n} = F + C(f) \end{array} \tag{3.33}$$

Regarding the $n(R)$-locus two conflictive effects can be eyeballed from the second equation of (3.33) which originate from the interaction between attention competition and economic competition. First, as less attentive consumers compare fewer alternatives this loosens price competition which leads to higher equilibrium prices and higher revenues (see lemma 3.5). Hence by the revenue effect we would expect profits and thus n to increase under limited attention. Second, because getting attention is of more value to firms, the competition for attention is intensified ($f'(R) < 0$) and attention costs increase. Thus the cost effect suggests n to decrease under limited attention. Manipulation of (3.33) shows that a single equation determines $n(R)$:

$$\frac{\Delta\tilde{V}(y(R))}{n^2\eta}(n(\eta-1)+R)=F \tag{3.34}$$

Using the implicit function theorem on (3.34) we can deduce that

$$sign(n'(R)) = sign\left(R + \left(R + n\left(\eta - 1\right)\right)\varepsilon_v\right) \tag{3.35}$$

with $\tilde{v}(R) \equiv \tilde{V}(y(R))$ and $\varepsilon_v \equiv \frac{\tilde{v}'(R)R}{\tilde{v}(R)} < 0$.[33] Expression (3.35) nicely confronts the two conflictive effects with each other. We see that the elasticity η and ε_v are important determinants of the curve $n(R)$. I first discuss the impact of ε_v. If $\varepsilon_v \approx 0$ e.g. because price competition is such that equilibrium prices respond only weakly to a change of R then $n'(R) > 0$. In this case less attention increases the revenue from the attention sets only weakly. At the same time competition for the sets is increased which implies higher costs of attention. As the additional costs outweigh the additional revenue gains profits decrease which leads to a smaller market. In the absence of price competition ($V(y, y, R) = v/R$) we unambiguously get $n'(R) > 0$ ($\varepsilon_v = 0$ because in this case $y'(R) = 0$). In order for the revenue effect to dominate the cost effect if consumer attention declines, the value of the attention set must respond sufficiently elastic to R. If this is the case (e.g. because commodities are strong substitutes and hence economic competition is intense, see section 3.4.2.3 below) then limited attention implies larger markets. To explain the impact of η on $n(R)$ note that a high value of η means that a change of f implies a strong change of marginal costs which reduces the reagibility of f. In this sense η controls the elasticity of attention competition: a high value of η means that f reacts less responsive to any exogenous change. To see this use (3.33) to find

$$f = \left(\frac{n-R}{n^2\theta\eta}\tilde{v}(R)\right)^{\frac{1}{\eta}} = \left(\frac{\tau}{\eta}\right)^{\frac{1}{\eta}} \qquad \tau \equiv \frac{n-R}{n^2\theta}\tilde{v}(R) \tag{3.36}$$

Hence $f'(\tau)\tau/f(\tau) = 1/\eta$. A higher value of η means that the equilibrium f does not react strongly to a change of R which also means that attention costs do not change much.

[33] The expression is negative because $\tilde{v}'(R) = \tilde{V}'\left(y(R)\right)y'(R)$ and $y'(R) < 0$ but $\tilde{V}'(y(R)) > 0$.

Thus the revenue effect is more likely to dominate the cost effect if η is large. Moreover, we can deduce from (3.34) that $n'(\eta) > 0$ which by (3.33) means that equilibrium costs $C(f)$ must decrease in η. From (3.34) we see that

$$\lim_{\eta\to\infty} n(\eta) = \frac{\Delta\tilde{V}(y)}{F}$$

Similarly, if $\theta = 0$ such that attention competition is free we get $f^* \to \infty$ and $n = \Delta\tilde{V}(y)/F$. In both cases equilibrium profits are only determined by the value of the attention set. As in both cases $C(f) = 0$ this can be thought of an upper bound on the market size (for given R).

To summarise we note that if the economic competition is weak ($\varepsilon_v \approx 0$), e.g. because of weak substitutes, then firms cannot extract much additional revenue from a decline of attention and the cost effect tends to dominate which implies smaller markets if consumers are less attentive. If however the economic competition is strong or attention competition is inelastic (high η) then less attentive consumers might imply higher profits and larger markets.

3.4.2.3 The comparative statics of the ACF-CES example

We can use the general results of section 3.4.2.2 to discuss the comparrative statics of the ACF-CES example considered earlier. Suppose the parameter values are such that a free-entry equilibrium with $n > R$ occurs endogenously. From section 3.3.3 we know that a single symmetric SPE exists in this example. As we use the ACF we immediately know from the previous section that $f'(\Delta) > 0$, $f'(R) < 0$ and $f(F)$ is non-monotonic (hump-shaped). Also $n'(\Delta) > 0$ and $n'(F) < 0$ follow directly from the last section. Because in the CES-example we have $V(y, y, R) = \frac{y-c}{y}\frac{v}{R}$ (see (3.31)) we can use (3.35) to examine $n'(R)$ in the ACF-CES case. It is straightforward to show that $\tilde{v}(R) = \frac{vR}{1+\sigma(R-1)}$ and $\varepsilon_v = \frac{(1-\sigma)}{1+(R-1)\sigma} \in (-1, 0)$. Hence for $\eta = 1$ we get $sign(n'(R)) = sign\,(R(1 + \varepsilon_v)$ and thus $n'(R) > 0$. In this case less attentive consumers imply a very fierce attention competition that involves high attention costs and leads to exit. However, if η is large we could also get $n'(R) < 0$, i.e. the gains from consumer inattention more than cover the increased attention costs. This leads to firm entry. Numerical evaluations of the ACF-CES example confirm that, depending on the parameter constellation[34], both cases may occur as figure 3.7 suggests. Finally, it is clear that in the ACF-CES example consumer budget has the

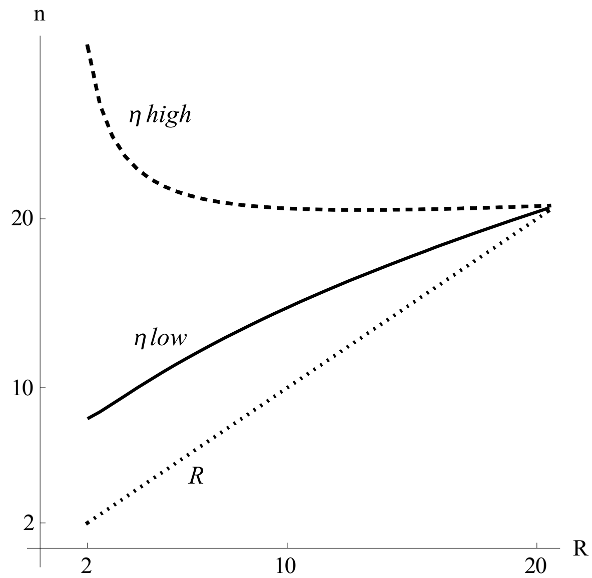

Figure 3.7: The ACF-CES example

same comparative statics as Δ, i.e. $y'(v) = 0$, $f'(v) > 0$ and $n'(v) > 0$. A larger value of σ means that commodities are stronger substitutes and hence the economic competition is more intense. It can be shown that $y'(\sigma) < 0$, $f'(\sigma) < 0$ and $n'(\sigma) < 0$ (see 3.7.6, part III).

3.5 Summary and comparison to the literature

I have presented a model of simultaneous strategic attention-price competition. This is the first contribution that develops a game-theoretic setting which allows to combine attention competition with economic competition. The competition for attention is modelled as a contest where the "loudest" firm has the highest chance of perception. In the free-entry equilibrium of the symmetric price-attention game prices, attention effort levels and the number of active firms are endogenously determined. I derive general conditions which assert that only one symmetric equilibrium exists and show that my main example of the attention contest, the ACF, satisfies these properties. In my model firms can influence the

[34]It can be shown that in the CES case we have $\varepsilon_v'(\sigma) < 0$ which means that if the varieties are stronger substitutes (larger σ) then, as suggested in the main text, $n'(R) < 0$ is more likely to occur.

chance of perception which depends on their own effort level but also on the effort level of all competitors and is endogenously determined. Competition for attention emerges only if the equilibrium number of firms (information senders) is higher than the exogenous attention threshold R. This is a similarity to the macroeconomic attention model of Falkinger (Falkinger (2008)), where an information-rich economy occurs if and only if total signal exposure at the consumer level exceeds a given perceptional threshold τ_0. The important difference between the two models, besides the fact that my approach is a game-theoretic one, lies in the way how attention competition is introduced. In his model a firm must send at a minimum signal strength σ_{min} in order to be perceived. Every firm that sends at σ_{min} is perceived by all consumers who receive information of the firm. This threshold value is exogenous to the single firm but endogenously determined in equilibrium and proportional to total signal exposure of a consumer. Sending at σ_{min} requires the firms to pay a fixed cost that depends positively on σ_{min}. That is, the fixed costs of attention increases in the minimal signal strength that asserts perception.

One prediction of his model is that, in an information-rich economy and if every firm is restricted to producing one commodity, if the firms can reach all consumers[35] by sending at σ_{min}, then the measure of active firms (T) is a measure of the attention parameter τ_0 because in this case $T = \tau_0$. This is completely different in my model as i) an attention equilibrium requires that there are more information senders than the people can perceive ($n > R$) and ii) as we have seen we may have $n'(R) > 0$ but also $n'(R) < 0$ can occur. The reason why a negative relationship between the attention parameter R and n can occur in my model follows only from the fact prices also depend on the attention parameter. Actually, $y'(R) < 0$ is a *necessary* condition for $n'(R) < 0$. Hence my model predicts that the number of active firms is not a suitable measure for people's attention threshold. Additionally, in my model the equilibrium price reflects limited attention which helps to explain why e.g. online prices have not dropped as much as expected after the introduction of the Internet - a fact that search models and models of informative advertising have significant difficulties to explain.

[35]In his model this requires that $r = R$.

3.6 Appendix A: On uniqueness and stability

In section 3.3 I have presented the assumptions on the value function V and the probability function p which exclude the possibility of multiple symmetric equilibria in the symmetric price-attention game. I have shown that the ACF satisfies all assumptions on p.

In this section I ask the more ambitious question whether there can be asymmetric equilibria in the symmetric game. Intuitively, the interaction of economic competition with attention competition suggests reasons why we could expect asymmetric "specialisation" equilibria to exist. We can imagine some firms to specialise in the economic competition. These firms set a low price and do not advertise a lot (i.e. choose a low attention effort level). The other firms specialise in attention competition: they choose a high price and high attention levels. A firm in the first group knows that its chance of perception is smaller than the chance of an attention specialist but i) it needs less revenue to cover attention costs and ii) it may earn more revenue than an attention specialist from those attention sets both are in. The firms in the second group are in more attention sets which could cover their higher attention expenditure.

I now derive the conditions that exclude the possibility of asymmetric equilibria in the price-attention game. To achieve this goal we cannot work with the symmetric opponent form of the profit function. But because of the complex nature of the price-attention game (every profit function has his own "context" B_j, see (3.5)) it is clear that standard approaches to uniqueness such as the univalence approach or the index theorem (see chapter 5.3.1) are not feasible because we would have to evaluate at $(2N) \times (2N)$-matrix! Fortunately, there exists an approach to uniqueness that separates between the possibility of multiple symmetric equilibria and the possibility of asymmetric equilibria. This approach is developed throughout chapter 5. The proof of the inexistence of asymmetric equilibria in the symmetric price-attention game in this section is an application of this methodology. Further, I briefly discuss stability of the symmetric equilibrium under iterative adjustments.

The proof is organized as follows. First, I consider a reduced version of the game where the attention distribution is given exogenously but prices are chosen simultaneously. I show that, under certain assumption, there only is one equilibrium in this pricing game, the symmetric equilibrium, where all firms choose the same price y independent of the attention distribution. Then it suffices to find conditions that assert that the pure attention game for given and identical prices y only has one equilibrium, the symmetric equilibrium.

3.6.1 Uniqueness of the symmetric equilibrium

First, I state the basic assumptions regarding price competition that are maintained throughout appendix A. Let $S_y \equiv [c, y^{max}]$ as before. Suppose that the attention distribution P as defined by (3.2) is exogenously fixed with the property that $P_A > 0$ for all $A \in \mathcal{A}$. A firm can only choose its price y_j. I consider the following profit function:

$$\Pi(y_j) = \sum_{A \in B_j} P_A V^j(A) \Delta \tag{3.37}$$

Suppose that all n firms simultaneously and non-cooperatively choose their price according to (3.37). Then $\left(n, S_y^n, \Pi\right)$ is an n-player pricing game. I impose the following assumptions on the value functions $V^j(Y_A)$ with $A \in B_j$:

Assumption 3.7. *Let $A \in B_j \subset \mathcal{A}$ and $|A| = z > 1$. Then*

a) For all $A \in B_j$ and $j = 1, ..., n$: $V^j(Y_A) = V^{\sigma(j)}(Y_{\sigma(A)})$, where σ is a permutation of A. $V^j(Y_A) > 0$ if $y_j = y_g = y \in Int(S_y)$ for all $g \in A$.

b) $V^j(Y_A) \in C^2\left(Int(S_y^z), \mathbb{R}\right)$, $V^j(Y_A)$ is strongly quasiconcave in y and $\frac{\partial V^j(Y_A)}{\partial y_g} > 0$ for $g \in A$.

c) $\left.\frac{\partial V^j(Y_A)}{\partial y}\right|_{y=c} > 0$ and $\left.\frac{\partial V^j(Y_A)}{\partial y}\right|_{y=y^{\max}} < 0$.

d) For $y_g \in (c, y^{\max})$ with $g \neq j$:

$$\sum_{A \in B_j} P_A \frac{\partial V^j(Y_A)}{\partial y_j} = 0 \quad \Rightarrow \quad \sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_1 y_g} > 0$$

The last assumption means that prices are strategic complements. In case of an equal distribution of attention, i.e. $P(A) = P(A') = \begin{pmatrix} n \\ R \end{pmatrix}^{-1}$ for all $A, A' \in \mathcal{A}$, the symmetric opponents form of (3.37) is

$$\Pi(y) = \begin{cases} \frac{R}{n} V(y, \bar{y}, R)\Delta & R < n \\ V(y, \bar{y}, n)\Delta & R \geq n \end{cases} \tag{3.38}$$

To see this in the case where $R < n$ note that $|B_j| = \begin{pmatrix} n-1 \\ R-1 \end{pmatrix}$. Then if $y_g = \bar{y}$ for any $g \neq j$ we have

$$\sum_{A \in B_j} P_A V^j(Y_A)\Delta = \frac{1}{\begin{pmatrix} n \\ R \end{pmatrix}} \begin{pmatrix} n-1 \\ R-1 \end{pmatrix} V(y, \bar{y}, R)\Delta = \frac{R}{n} V(y, \bar{y}, R)\Delta$$

Lemma 3.5. *Let $P(A) = P(A')$ for all $A, A' \in \mathcal{A}$ and take assumption 3.7 as well as condition* (3.18) *to be satisfied. Then the interior symmetric equilibrium $y^* \in (c, y^{max})$ is the unique equilibrium of the pricing game.*

Proof: Appendix B (3.7.12)

Note that because $\frac{\partial}{\partial y}(V_1(y, y, z)) = V_{11}(y, y, z) + V_{12}(y, y, z)$ the inequality in condition (3.18) can be rewritten as $-V_{11}(y, y, z) > V_{12}(y, y, z)$. If even

$$V_1(y^*, y^*, z) = 0 \;\Rightarrow\; -V_{11}(y^*, y^*, z) > |V_{12}(y^*, y^*, z)| \tag{3.39}$$

holds then the symmetric equilibrium y^* is locally contraction-stable[36] (stable under iterative adjustments).

Lemma 3.5 implies that if $R \geq n$ the general proof of uniqueness in the symmetric price-attention game is already finished because there cannot be an equilibrium other than (y^*, f) with $f = 0$ and y^* defined by $V_1(y^*, y^*, n) = 0$.

From now on I always set $n > R$. Suppose that the attention distribution function P is still exogenously given but $P_A > 0$ may be different for different $A \in \mathcal{A}$. An interior

[36] See theorem 5.4.

equilibrium $(y_1, ..., y_n)$ of the pricing game satisfies

$$\begin{pmatrix} \sum\limits_{A\in B_1} P_A \frac{\partial V^1(Y_A)}{\partial y_1} \\ \vdots \\ \sum\limits_{A\in B_n} P_A \frac{\partial V^n(Y_A)}{\partial y_n} \end{pmatrix} \equiv D_y\left(y_1, ..., y_n\right) = 0 \tag{3.40}$$

The following proposition shows that under a further assumption the only interior equilibrium price vector is symmetric and independent of the attention distribution.

Proposition 3.6. *Suppose that $n > R$ and $P_A > 0$ for all $A \in \mathcal{A}$. Then under assumption 3.7 and condition* (3.18) *there exists a single symmetric equilibrium y^* that is independent of the distribution of P_A. Further, if for any feasible y_g with $g \neq j$ and any $A' \in B_j$ with $j, g \in A'$*

$$\sum_{A\in B_j} P_A \frac{\partial V^j(Y_A)}{\partial y_j} = 0 \Rightarrow -\frac{\partial^2 V^j\left(Y_{A'}\right)}{\partial y_j^2} > (R-1)\left|\frac{\partial^2 V^j\left(Y_{A'}\right)}{\partial y_j \partial y_g}\right| \tag{3.41}$$

holds then y^* is the unique equilibrium of the pricing game.

Proof: Appendix B (3.7.13)

Note that if the inequality in condition (3.41) holds even without the requirement that $\sum\limits_{A\in B_j} P_A \frac{\partial V^j(Y_A)}{\partial y_j} = 0$ then uniqueness follows directly from the contraction mapping theorem (see Vives (1999), p. 47).[37]

Now I return to the symmetric price-attention game from the main text.

Proposition 3.7 (Sufficient conditions for uniqueness)**.** *Suppose $n > R$ and assumptions 3.4 and 3.7 as well as* (3.41) *and* (3.19) *are satisfied. If for all $A \in B_j$, $f_1, ..., f_n \in (0, \infty)$ and any fixed player $g \neq j$ the condition*

$$-\frac{\partial^2 P_A(\mathcal{F})}{\partial f_j^2} + \frac{\partial^2 P_A(\mathcal{F})}{\partial f_j \partial f_g} > 0 \tag{3.42}$$

holds then the symmetric equilibrium (y, f) determined by (3.16) *is the unique equilibrium of the symmetric price-attention game.*

[37] In the CES example it can be shown by applying proposition (3.2) for e.g. $\sigma = 2$, $R = 2$ and $n = 3$ that in this case the joint best-response function is not a global contraction but condition (3.41) holds resulting in a unique price-equilibrium.

Proof: Appendix B (3.7.14)

Note that if all presuppositions of proposition 3.7 are satisfied for any $n \geq 2$ and (3.27) and (3.28) hold then there only is one equilibrium (y, f, n) in the free-entry game and the equilibrium is symmetric.

From the main text we already know that the ACF does not generate multiple symmetric equilibria. But does the ACF also satisfy (3.42)?

Consider (3.14) and suppose firm j did not get the first draw. Then one of firm j's competitors must have succeeded in doing so. The aggregate mass of the balls remaining in the urn has changed conditional on the ball drawn out. Suppose for example that the ball of firm $g \neq j$ has been drawn. Then conditional on this event firm j's chance of getting the next draw is $\frac{f_j}{\sum_{k=1}^{n} f_k - f_g}$. We can use such a line of reasoning to calculate the probabilities P_A for all $A \in \mathcal{A}$. To illustrate this, suppose $R = 2$ and $n = 3$. Then $\mathcal{A} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. Let $A = \{1,2\}$. Then $P_A(\mathcal{F}) = P_A(f_1, f_2, f_3)$ and

$$P_A(f_1, f_2, f_3) = \frac{f_1 f_2}{f_1 + f_2 + f_3}\left(\frac{1}{f_2 + f_3} + \frac{1}{f_1 + f_3}\right)$$

Corollary 3.3. *Let $R = 2$. The ACF satisfies condition*(3.42).

Proof: Appendix B (3.7.15)

Remark: The procedure of the proof of 3.3 can be applied to show that (3.42) is satisfied also if $R = 3$. I conjecture that (3.42) holds for any $R > 2$ under the ACF.

3.6.2 An example

I now provide an example of a function V^j that satisfies assumption 3.7 and condition (3.41). Let j denote the representative firm. Then for any $A \in B_j$ let

$$V^j(Y_A) = (y_j - c)x^j(Y_A)$$

with

$$x^j(Y_A) = \max\left\{0, \frac{1}{1 + (z-1)\gamma}\left(1 - \frac{1 + (z-2)\gamma}{1-\gamma} y_j + \frac{\gamma}{1-\gamma}\sum_{i \in A: i \neq j} y_i\right)\right\} \tag{3.43}$$

where $\gamma \in (0,1)$ controls the degree of substitutability[38] between perceived commodities and $z = min\{R, n\}$. This demand function is often used in oligopolistic theory and IO models and can be microfounded by assuming a quasilinear upper tier utility function and quadratic subutility (see e.g. Vives (1999) p. 145-146). For simplicity, I set $c = 0$ and $\Delta = 1$.

Theorem 3.1. *Suppose $S_y = [0,1]$. Then with (3.43) the symmetric equilibrium $y^* \in (0,1)$ of the pricing game is unique and independent of the attention distribution.*

Proof:

In step 1 I show that assumption 3.7 is satisfied. In step 2 I show that (3.41) is satisfied.

Step 1: Assume $j \in A \in B_j$.

a) Permutation-invariance is obvious. Let $y_j = y_g = y \in (0,1)$ for all $g \in A$. Then $V^j(Y_A) = \frac{(1-y)y}{1+(z-1)\gamma} > 0$.

b) We have $\frac{\partial^2 V^j(Y_A)}{\partial y_j^2} = -2\frac{1+(z-2)\gamma}{(1-\gamma)(1+(z-1)\gamma)} < 0$ because $z > 1$ and $\gamma < 1$. Hence $V^j(Y_A)$ is strongly concave and thus also strongly quasiconcave.

c) We have

$$\left.\frac{\partial V^j(Y_A)}{\partial y_j}\right|_{y=0} = \frac{1 + \frac{\gamma}{1-\gamma}\sum\limits_{i\in A: i\neq j} y_i}{1+(z-1)\gamma} > 0$$

and

$$\left.\frac{\partial V^j(Y_A)}{\partial y_j}\right|_{y=1} = -\frac{1+(2z-3)\gamma - \gamma\sum\limits_{i\in A: i\neq j} y_i}{(1-\gamma)(1+(z-1)\gamma)}$$

Thus $\left.\frac{\partial V^j(Y_A)}{\partial y_j}\right|_{y=1} < 0$ because $1 + \gamma(z-2) > 0$.

d) Because $\frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g} = \frac{\gamma}{(1-\gamma)(1+(z-1)\gamma)} > 0$ for any $g \neq j$ with $g \in A$ the game is strictly supermodular in prices. Hence prices are strategic complements.

Step 2:

Assume $g \in A$. Then

$$\begin{aligned} -\frac{\partial^2 V^j(Y_A)}{\partial y_j^2} > (z-1)\left|\frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g}\right| &\Leftrightarrow 2\frac{1+(z-2)\gamma}{(1-\gamma)(1+(z-1)\gamma)} > \frac{\gamma(z-1)}{(1-\gamma)(1+(z-1)\gamma)} \\ &\Leftrightarrow 2+\gamma(z-3) > 0 \end{aligned}$$

[38] $\gamma \to 1$ means that commodities are perfect substitutes whereas $\gamma \to 0$ means that commodities are independent.

which holds as $z > 1$ and $\gamma < 1$. Hence condition (3.41) from proposition 3.6 holds and so does condition (3.18). This implies the existence of a unique interior equilibrium for any $n, R > 1$ that is independent of the attention distribution. Moreover, the equilibrium is symmetric.

■

If $R = 2$ we know from corollary 4 that if we use the ACF to determine the attention probability distribution the price-attention game only has a unique symmetric equilibrium.

3.6.3 Stability

This section provides conditions which imply that the symmetric equilibrium is a local contraction.

Proposition 3.8. *The symmetric equilibrium (y, f) of the price-attention game is locally contraction-stable if $V(y, \bar{y}, R)$ is a local contraction at y (condition (3.39) holds) and for $n > R$*

$$\Delta \frac{p_1(f, f, n, R)V_2(y, y, R) + |p_{12}(f, f, n, R)|\, V(y, y, R)}{-\left(p_{11}(f, f, n, R)\Delta V(y, y, R) - C''(f)\right)} < 1 \tag{3.44}$$

Proof: Follows from theorem 5.4.

■

Corollary 3.4. *Suppose $n > R$ and $V_2(y, y, R) = 0$ and let $C(f) = \theta f^{\eta}$ where $\eta \geq 2$ and $p(f, \bar{f}, n, R)$ is given by the ACF. Then the symmetric equilibrium is locally contraction-stable.*

Proof: Appendix B (3.7.16)

Note that e.g. with $\eta = 1$ the contraction condition (3.44) need not hold generally for the ACF. In any case we require $V_2(y, y, R)$ to be small. Intuitively, the contraction condition rather holds with higher η as then attention competition is less elastic (see (3.36)) and hence firms only respond weakly to small deviations around the symmetric equilibrium.

3.7 Appendix B

3.7.1 Properties of the ACF

Lemma 3.6. *There exists a continuously differentiable extension to* (3.13) *which is given by*

$$p^C(f, f, n, R) = 1 - \frac{\Gamma(n)}{\Gamma(n-R)} \frac{\Gamma\left(\frac{f}{\bar{f}} + n - R\right)}{\Gamma\left(\frac{f}{\bar{f}} + n\right)} \tag{3.45}$$

where $\Gamma(x)$ *denotes the Gammafunction evaluated at* x *and* R *and* n *are real numbers with* $R < n$.

$$p_1^C(f, f, n, R) = \frac{n-R}{nf} \left(\psi\left(1+n\right) - \psi\left(1+n-R\right)\right) \tag{3.46}$$

where $\psi(.)$ *is the digamma function. If* R_0 *is an integer then*

$$p_1^C(f, f, n, R_0) = \frac{n-R_0}{nf} \sum_{i=1}^{R_0} \frac{1}{1+n-i}$$

and $p_1^C(f, f, n, R_0)$ *is strictly concave in* R *around* R_0.

Proof:

First note that

$$\prod_{i=1}^{R} \left((n-i)\bar{f}\right) = \bar{f}^R \frac{(n-1)!}{(n-R)!}(n-R) \tag{3.47}$$

The Gamma-function $\Gamma(x)$, which is continuously differentiable on $x \in \mathbb{R}^+$, has the properties that $n! = \Gamma(n+1)$ if n is a positive integers and $\Gamma(x+1) = x\Gamma(x)$. Hence (3.47) becomes

$$\bar{f}^R \frac{\Gamma(n)}{\Gamma(n-R)}$$

Equivalently,

$$\begin{aligned} \prod_{i=1}^{R} \left(f + (n-i)\bar{f}\right) &= \bar{f}^R \prod_{i=1}^{R} \left(\frac{f}{\bar{f}} + n - i\right) \\ &= \bar{f}^R \frac{\Gamma\left(\frac{f}{\bar{f}}+n\right)}{\Gamma\left(\frac{f}{\bar{f}}+n-R\right)} \end{aligned} \tag{3.48}$$

Hence

$$p^C(f,f,n,R) = 1 - \frac{\Gamma(n)}{\Gamma(n-R)} \frac{\Gamma\left(\frac{f}{f}+n-R\right)}{\Gamma\left(\frac{f}{f}+n\right)}$$

which is continous for $R < n$ and $P(f) \overset{R\to n^-}{\longrightarrow} 1$ as $\Gamma(n-R) \overset{R\to n^-}{\longrightarrow} \infty$. As the derivation makes clear we have $p(f,\bar{f},n,R) = p^C(f,\bar{f},n,R)$ whenever R and n are integers. Differentiation of (3.45) at $f = \bar{f} > 0$ then yields (3.46). If R_0 is an integer we have

$$\psi\left(1+n\right) = \psi\left(1+n-R_0\right) + \sum_{i=1}^{R_0} \frac{1}{1+n-i}$$

as also the digamma function is recursive with $\psi(1+x) = \psi(x) + \frac{1}{x}$. Hence

$$p_1^C(f,f,n,R_0) = \frac{n-R_0}{nf} \sum_{i=1}^{R_0} \frac{1}{1+n-i}$$

which corresponds to (3.20). But know is is formally possible to take the derivative and

$$p_{14}^C(f,f,n,R_0) = \frac{1}{nf}\left((n-R_0)\frac{1}{(1+n-R_0)^2} - \sum \frac{1}{1+n-i}\right)$$

But

$$p_{14}^C(f,f,n,R_0) - \frac{p_1^C(f,f,n,R_0)}{R_0} = \frac{n-R_0}{(1+n-R_0)^2} - \frac{n}{R_0} \sum_{i=1}^{R_0} \frac{1}{1+n-i}$$

which is negative as $\frac{n-R_0}{(1+n-R_0)^2} < 1$ but $\sum_{i=1}^{R_0} \frac{1}{1+n-i} = \frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{1+n-R_0} > \frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n} = \frac{R_0}{n}$. Hence $p_1^C(f,f,n,R_0)$ is strictly concave in R around R_0.[39]

■

Lemma 3.7. *With the ACF based on* (3.20) *the following properties hold:*

a) $\frac{p_1(f,f,n,R)}{n} + p_{13}(f,f,n,R) > 0$

b) $\frac{p''(f,f,n,R)}{p_1(f,f,n,R)} < p_{13}(f,f,n,R)\frac{n^2}{R}$

c) $\frac{p''(f,f,n,R)}{p_1(f,f,n,R)} < -\frac{n}{R} p_1(f,f,n,R)$

[39] It is an easy numerical excersise to show that this result generalises for an arbitrary real-valued R_0.

d) *$p_1(f,f,n,R)$ for given $f>0$ is a hump-shaped function of n.*

where $p''(f,f,n,R) \equiv \frac{\partial p_1(f,f,n,R)}{\partial f}$.

Proof:

a) As

$$p_{13}(f,f,n,R) = \frac{1}{nf}\left(\frac{R}{n}\sum_{i=1}^{R}\frac{1}{1+n-i} - (n-R)\sum_{i=1}^{R}\left(\frac{1}{1+n-i}\right)^2\right) \tag{3.49}$$

we have

$$\frac{p_1}{n} + p_{13} > 0 \;\Leftrightarrow\; \sum_{i=1}^{R}\frac{1}{1+n-i} - (n-R)\sum_{i=1}^{R}\left(\frac{1}{1+n-i}\right)^2 > 0 \tag{3.50}$$

which holds because

$$\sum_{i=1}^{R}\frac{1}{1+n-i} - (n-R)\sum_{i=1}^{R}\left(\frac{1}{1+n-i}\right)^2 = \sum_{i=1}^{R}\left(\frac{1+R-i}{(1+n-i)^2}\right) > 0$$

b) From (3.20) we see that $\frac{p''}{p_1} = -\frac{1}{f}$. Then using (3.49) gives

$$\frac{p''}{p_1} < p_{13}\frac{n^2}{R} \;\Leftrightarrow\; -\frac{R}{n^2} < \frac{R}{n^2}\sum_{i=1}^{R}\frac{1}{1+n-i} - \frac{(n-R)}{n}\sum_{i=1}^{R}\left(\frac{1}{1+n-i}\right)^2$$

Reformulation gives

$$\frac{R}{n}\left(\sum_{i=1}^{R}\frac{1}{1+n-i} + 1\right) - (n-R)\sum_{i=1}^{R}\left(\frac{1}{1+n-i}\right)^2 > 0$$

or equivalently

$$\frac{R}{n}\left(\sum_{i=1}^{R}\frac{1}{1+n-i} + 1\right) - (n-R)\sum_{i=1}^{R}\left(\frac{1}{1+n-i}\right)^2 + \sum_{i=1}^{R}\frac{1}{1+n-i} - \sum_{i=1}^{R}\frac{1}{1+n-i} > 0$$

But then because of (3.50) we only need to show

$$\frac{R}{n}\left(\sum_{i=1}^{R}\frac{1}{1+n-i}+1\right)-\sum_{i=1}^{R}\frac{1}{1+n-i}>0$$

or

$$R-\sum_{i=1}^{R}\frac{1}{1+n-i}(n-R)>0 \tag{3.51}$$

which is true because

$$R-\sum_{i=1}^{R}\frac{1}{1+n-i}(n-R)=\sum_{i=1}^{R}\left(1-\frac{n-R}{1+n-i}\right)=\sum_{i=1}^{R}\left(\frac{1+R-i}{1+n-i}\right)>0$$

c) Because $\frac{p''}{p_1}=-\frac{1}{f}$ we have

$$\frac{p''}{p_1}<-\frac{n}{R}p_1 \;\Leftrightarrow\; 1>\frac{n-R}{R}\sum_{i=1}^{R}\frac{1}{1+n-i}$$

or equivalently

$$R-(n-R)\sum_{i=1}^{R}\frac{1}{1+n-i}>0$$

which holds because of (3.51).

d) Let $n>R$ and use (3.49) to find

$$\begin{aligned} sign\,(p_{13}(f,f,n,R)) &= sign\left(\sum_{i=1}^{R}\left(2R-n+\tfrac{R}{n}(1-i)\right)\right)\\ &= sign\left(\underbrace{(2R-n)+\frac{R}{n}-\frac{R(R+1)}{2n}}_{=\psi(n)}\right)\end{aligned}$$

But $n>R$ implies that $\psi'(n)=-1-\frac{R}{n^2}+\frac{R(R+1)}{2n^2}<0$. Moreover, we have $\psi(R)=\frac{R+1}{2}>0$ and $\psi(2R)=\frac{1}{2}-\frac{(R+1)}{4}<0$ because $R>1$. These findings imply that $p_1(f,f,n,R)$ must be hump-shaped in n.

■

3.7.2 Proof of proposition 3.1

Use (3.3) to find

$$\sum_{j\in I} p_{ij} = \sum_{j\in I}\left(\sum_{A\in\mathcal{A}} P_{iA}\, 1\,[j\in A]\right) = \sum_{A\in\mathcal{A}} P_{iA} \sum_{j\in I} 1\,[j\in A]$$
$$= \sum_{j\in I} 1\,[j\in A] = z$$

■

3.7.3 Proof of lemma 3.1

The first part follows directly from (3.3). If $R \geq n$ then $p_j = 1$ for any active firm because there only is one attention set, namely I itself, which is perceived with probability one. Now suppose $n > R$. Then (3.3) implies $p_j = \sum_{A\in B_j} P_A(\mathcal{F})$ which gives $p_j = p_j(f_j, f_{-j})$. Now suppose that $p_j < 1$. Then $\sum_{A\in B_j} P_A(\mathcal{F}) < 1$ which by assumption 3.1 implies that $P_A(\mathcal{F})$ increases in f_j for all $A \in B_j$. But this implies that $p_j(f_j, f_{-j})$ increases in f_j.

■

3.7.4 Proof of lemma 3.2

Suppose $n > R$. Because $f_g = \bar{f}$ for any $g \neq j$ we have $P_A(\mathcal{F}) = P_{A'}(\mathcal{F})$ for $A, A' \in B_j$. Let $s \equiv |B_j|$ and suppose $j \in A$. Then $\sum_{A\in B_j} P_A(\mathcal{F}) = sP_A(\mathcal{F})$ for $A \in B_j$. Lemma 3.1 and (3.3) imply that $sP_A(\mathcal{F}) = p(f, \bar{f}, n, R)$. Hence

$$\sum_{A\in B_j} P_A(\mathcal{F}) V^j(Y_A)\Delta = V(y, \bar{y}, R)\Delta \sum_{A\in B_j} P_A(\mathcal{F})$$
$$= p(f, \bar{f}, n, R) V(y, \bar{y}, R)\Delta$$

If this is used in (3.5) we get (3.9). If $R \geq n$ then $p_j = 1$ and $V^j(Y_A) = V(y, \bar{y}, n)$ which then implies (3.10).

■

Remark: Hence if we know $p(f, \bar{f}, n, R)$ then we also know $P_A(\mathcal{F})$ for any $A \in B_j$. But then we also can find $P_{A'}(\mathcal{F})$ for any $A' \notin B_j$. To see this let $B_{-j} \equiv \{A' \in \mathcal{A} : j \notin A'\}$.

Then we have $P_{A'}(\mathcal{F}) = P_{A''}(\mathcal{F})$ for any $A', A'' \in B_{-j}$. Then

$$\begin{aligned}\sum_{A' \in B_{-j}} P_{A'}(\mathcal{F}) &= 1 - \sum_{A \in B_j} P_A(\mathcal{F}) \\ &= 1 - sP_A(\mathcal{F}) = 1 - p(f, \bar{f}, n, R)\end{aligned}$$

3.7.5 Proof of proposition 3.2

Let $n > R$. Define

$$G(f) \equiv \prod_{i=1}^{R} g(f, i) \tag{3.52}$$

with

$$g(f, i) \equiv \frac{(n-i)\bar{f}}{f + (n-i)\bar{f}}$$

a) Obviously $G(f) \in [0, 1]$ for $f, \bar{f} > 0$. Hence $p \in [0, 1]$. b) Relativity is obvious. c) With

$$-\frac{\partial G(f)}{\partial f} = -G(f) \sum_{i=1}^{R} \frac{\partial g(f, i)}{\partial f} \frac{1}{g(f, i)}$$

and

$$\begin{aligned}\tfrac{\partial g(f,i)}{\partial f} \tfrac{1}{g(f,i)} &= -\tfrac{(n-i)\bar{f}}{\left(f+(n-i)\bar{f}\right)^2} \tfrac{f+(n-i)\bar{f}}{(n-i)\bar{f}} \\ &= -\tfrac{1}{f+(n-i)\bar{f}}\end{aligned}$$

we get

$$p_1(f, \bar{f}, n, R) = -\frac{\partial G(f)}{\partial f} = G(f) \sum_{i=1}^{R} \frac{1}{f + (n-i)\bar{f}} > 0 \tag{3.53}$$

The same reasoning gives

$$p_2(f, \bar{f}, n, R) = -G(f) \sum_{i=1}^{R} \frac{\partial g(f, i)}{\partial \bar{f}} \frac{1}{g(f, i)}$$

But as

$$\frac{\partial g(f, i)}{\partial \bar{f}} = \frac{f(n-i)}{\left(f + \bar{f}(n-i)\right)^2} > 0$$

we have $p_2(f, \bar{f}, n, R) < 0$. Similarly, $p_3(f, \bar{f}, n, R) < 0$ as

$$\frac{\partial g(f, i)}{\partial n} = \frac{f\bar{f}}{\left(f + \bar{f}(n-i)\right)^2} > 0$$

Suppose $n \geq R+1$. The (strong) monotonicity of p in R can be seen from

$$G(f,R+1) - G(f,R) = -G(f,R)\left(\frac{f}{f+\bar{f}\,(n-(R+1))}\right) < 0$$

Finally, I show that $p_{11}(f,\bar{f},n,R) < 0$.

$$\begin{aligned}
p_{11}(f,\bar{f},n,R) &= -\tfrac{\partial^2 G(f)}{\partial f^2} \\
&= -\left(\tfrac{\partial G(f)}{\partial f}\sum_{i=1}^{R}\tfrac{\partial g(f,i)}{\partial f}\tfrac{1}{g(f,i)} + G(f)\sum_{i=1}^{R}\left(\tfrac{\partial^2 g(f,i)}{\partial f^2}g(f,i) - \left(\tfrac{\partial g(f,i)}{\partial f}\right)^2\right)\tfrac{1}{g(f,i)^2}\right) \\
&= -G(f)\left(\left[\left(\sum_{i=1}^{R}\tfrac{\partial g(f,i)}{\partial f}\tfrac{1}{g(f,i)}\right)^2 - \sum_{i=1}^{R}\left(\tfrac{\partial g(f,i)}{\partial f}\tfrac{1}{g(f,i)}\right)^2\right] + \sum_{i=1}^{R}\left(\tfrac{\partial^2 g(f,i)}{\partial f^2}g(f,i)\right)\tfrac{1}{g(f,i)^2}\right)
\end{aligned} \tag{3.54}$$

But as $\left(\sum_{i=1}^{R} a_i\right)^2 > \sum_{i=1}^{R} a_i^2$ for $a_i > 0$ and $\frac{\partial^2 g(f,i)}{\partial f^2} > 0$ the result follows immediately.

∎

3.7.6 The ACF-CES example: calculations

Part I:

In case of (3.11) $V_1(y,\bar{y},R) = 0$ implies

$$y^{\sigma}(R-1)(\sigma-1) = c\left(\bar{y}^{\sigma-1} + y^{\sigma-1}(R-1)\sigma\right) \tag{3.55}$$

which is equivalent to

$$y^{\sigma-1}(R-1)\left(y(\sigma-1) - c\sigma\right) = c\bar{y}^{\sigma-1}$$

which for $\bar{y} \geq c$ implies that $y > \frac{c\sigma}{\sigma-1}$ and hence also $y > c$. Applying the Implicit Function Theorem to (3.55) an rearranging gives

$$y'(\bar{y}) = c\frac{(y\bar{y})^{2-\sigma}}{\sigma(R-1)(y-c)} > 0$$

and

$$y'(R) = y\frac{y(1-\sigma)+c\sigma}{(R-1)(\sigma-1)\sigma(y-c)} < 0$$

Part II:

Suppose that $n > R$. Use $\bar{y} = y$ in (3.55). This gives (3.29). Moreover,

$$V(y, y, R) = (y - c)\frac{v}{yR} = \frac{v}{1 + (R - 1)\sigma} \tag{3.56}$$

Using this in (3.22) gives (3.30).

Part III:

$y'(R)$ follows directly from (3.29). From proposition 3.9 (see 3.7.11) we can deduce $sign(f'(\sigma)) = sign\left(V_{1\sigma}(p_{13} + \frac{p_1}{n}\right)$ and thus with the ACF $f'(\sigma) < 0$ follows because of $V_{1\sigma} = -\frac{(y-c)v(R-1)}{R^2y^2} < 0$. Similarly, we get $sign(n'(\sigma)) = sign\left(-V_{1\sigma}\left(np_1^2 + Rp'' - \omega\right)\right)$ and because of the ACF we have $n'(\sigma) < 0$.

3.7.7 Proof of proposition 3.3

Because the function $V(y, \bar{y}, z)$ is twice continuously differentiable in $y, \bar{y}$, the function $V_1(y, y, z)$ must be continuous in $y \in (c, y^{max})$ for any $z > 1$. Because of the boundary condition a) i) in assumption 3.5 the equation $V_1(y, y, z) = 0$ must then have a solution $y = y(z)$ for any $z > 1$ and $y \in (c, y^{max})$. Moreover, because of (3.18) the solution to $V_1(y, y, z) = 0$ must be unique for any given $z > 1$ (this is an index theorem result; see proposition 5.3 in chapter 5). Hence if $R \geq n$ then there exists exactly one symmetric equilibrium (y, f) with $f = 0$ and $y = y(n) \in (c, y^{max})$.

If $n > R$ repeating the previous argument shows that there exists a unique solution $y = y(R) \in (c, y^{max})$ to $V_1(y, y, R) = 0$. Because $p_1(f, f, n, R)$ is continuous for $f \in (0, \infty)$ (a consequence of assumption 3.4) and because of the boundary conditions a) ii) in assumption 3.5, $V(y, y, z) \in (0, \infty)$ and (3.7) the equation

$$\psi(f) \equiv p_1(f, f, n, R)V(y, y, R)\Delta - C'(f) = 0$$

must have a solution $f = f(n, R, \Delta) \in (0, \infty)$. Condition (3.19) together with (3.7) implies that $\psi(f)$ is strictly decreasing in $f \in (0, \infty)$ which means that the solution must be unique.

Consequently, there exists a unique vector (y, f) that solves (3.17).

(Second-order conditions) If $R \geq n$ then $V_{11}(y, y, n) < 0$ implies that second-order conditions of the representative firm's maximisation problem in (3.10) are satisfied at $y = y(n)$. Similarly, if $n > R$ then the second-order conditions of (3.9) evaluated at $y = y(R)$ and $f = f(n, R, \Delta)$ are satisfied because $\Pi_{11} = \frac{R}{n} V_{11}(y, y, R)\Delta < 0$, $\Pi_{22} = p_{11}(f, f, n, R)V(y, y, R)\Delta - C''(f) < 0$ and $\Pi_{12} = p_1(f, f, n, R)V_1(y, y, R)\Delta = 0$.

Thus the existence of exactly one symmetric equilibrium has been proved.

■

3.7.8 Proof of lemma 3.4

Let $n > R$. If we set $f = \bar{f} > 0$ then (3.52) gives $G(f) = \frac{n-R}{n}$. Using this in (3.53) gives (3.20). Moreover, (3.20) shows that the ACF satisfies assumption 3.5. Note that

$$\sum_{i=1}^{R} \frac{1}{1+n-i} = \sum_{j=0}^{R-1} \frac{1}{n-j} = H(n) - H(n-R) \tag{3.57}$$

where $H(x)$ is the x-th harmonic number. It is well known that the sequence $(H(x) - Ln(x))_{x \in \mathbb{N}_+}$ is monotonically decreasing and converges to the Euler-Mascheroni constant γ. Hence

$$H(n) - Ln(n) = \gamma + e(n)$$

where $e(n)$ is an error term with $\frac{de}{dn} < 0$. Thus the maximal error occurs at $n = 1$ and has $e(1) = 1 - \gamma < 1/2$. Because of the subtraction in (3.57) the constant γ cancels out. Further $e(n - R)$ is large if R is close to n. Thus the overall numerical error in setting $H(x) = Ln(x) + \gamma$ in (3.57), $e = e(n) - e(n - R)$ is bounded in absolute value by one and is small if n large and R small. Thus

$$\sum_{i=1}^{R} \frac{1}{1+n-i} \cong Ln(n) - Ln(n-R) = Ln\left(\frac{n}{n-R}\right)$$

By defining $x \equiv R/n$ we have

$$Ln\left(\frac{n}{n-R}\right) = Ln\left(\frac{1}{1-x}\right)$$

Applying a first-order Taylor expansion at $x_0 = 0$ then gives

$$Ln\left(\frac{1}{1-x}\right) = x + r_1\left(0,x\right)$$

with $|r_1\left(0,x\right)| \leq \frac{x^2}{1-x}$. Thus again if R is relatively small compared to n we can expect our approximation to perform numerically well. Thus the suggested approximation (3.21) is nummerically accurate whenever R/n is small.

■

3.7.9 Proof of proposition 3.4

Existence:

From (3.23) we see that profits are continuous in (y, f, n), because assumption 3.6 implies that $V(y, y, z)$ with $z = min\left\{R, n\right\}$ is continuous in n. Further we have $\Pi(\hat{y}, 0, 2) \geq 0$ by presupposition. $n \to \infty$ implies $n > R$. Hence $\lim_{n\to\infty} \Pi(y, f, n) = -F - C(f) < 0$. From proposition 3.3 we know that for any given $n \geq 2$ a solution (y, f) to (3.24) - (3.25) exists. Moreover, because of lemma 3.8 (see below) $(y, f) = (y(n), f(n))$ is a continuous function of n. Hence $\Pi(y(n), f(n), n)$ is a continuous function of n and consequently there must exist $n' \in [2, \infty)$ such that $\Pi(y(n'), f(n'), n') = 0$ which proves existence.

Uniqueness:

Suppose $n \geq> 2$ is given exogenously. Let $(y(n), f(n))$ denote the solution to (3.17). Because of proposition 3.3 and lemma 3.8 we now that $(y(n), f(n))$ is unique for any given n and that the vector $(y(n), f(n))$ is a continuously differentiable function of n if $n \neq R$. Define $\tilde{\Pi}(n) \equiv \Pi\left(y(n), f(n), n\right)$ where the function Π is given by (3.23). Then $\tilde{\Pi}(n)$ is a continuously differentiable function of n if $n \neq R$. But by the definition of $\tilde{\Pi}$ the vector $(y(n'), f(n'), n')$ is a symmetric equilibrium of the free entry game if and only if we have $\tilde{\Pi}(n') = 0$. If we have $\tilde{\Pi}'(n) < 0$ for all $n \neq R$ that satisfy $\tilde{\Pi}(n) = 0$ $(*)$ then there can be at most one $n \geq 2$ that solves $\tilde{\Pi}(n) = 0$. I now show that (3.28) is a sufficient condition for $(*)$.

Let $n < R$. Then

$$\tilde{\Pi}'(n) = \Delta \left(\underbrace{-\frac{V_2(y,y,n)V_{13}(y,y,n)}{\frac{\partial}{\partial y}\left(V_1(y,y,n)\right)}}_{<0} + \underbrace{V_3(y,y,n)}_{<0} \right) < 0$$

where $\tilde{\Pi}'(n)$ is evaluated at $\tilde{\Pi}(n) = 0$. Consequently, condition $(*)$ holds if $n < R$.

Now let $n > R$. Then

$$\tilde{\Pi}'(n) = -\frac{R}{n^2} - p_1(f,f,n,R)f'(n)$$

where $\tilde{\Pi}'(n)$ is evaluated at $\tilde{\Pi}(n) = 0$. But (see 3.7.10)

$$f'(n) = -\frac{p_{13}(f,f,n,R)\Delta V(y,y,R)}{\frac{\partial}{\partial f}\left(p_1(f,f,n,R)\right)\Delta V(y,y,R) - C''(f)}$$

This implies

$$\tilde{\Pi}'(n) = -\frac{R}{n^2} + p_1 \frac{p_{13}\Delta V}{\frac{\partial}{\partial f}\left(p_1\right)\Delta V - C''}$$

Then we have $\tilde{\Pi}'(n) < 0$ if

$$p_{13}\frac{n^2}{R} > \frac{\frac{\partial}{\partial f}\left(p_1\right)}{p_1} - \frac{C''}{\Delta V p_1} \tag{3.58}$$

Because $\frac{C''}{\Delta V p_1} \geq 0$ condition (3.28) is a sufficient condition for (3.58). Consequently, condition $(*)$ also holds if $n > R$.

Thus there only is one vector (y, f, n) that solves (3.24)-(3.26). Hence there only is one symmetric equilibrium.

■

Lemma 3.8. *Let $R, n > 1$. Suppose assumptions 3.5 and 3.6 are satisfied. Then the equilibrium vector (y, f) defined by (3.17) is a continuous function of n and continuously differentiable in n except at $n = R$.*

Proof: Because of proposition 3.3 $\exists!(y, f)$ that solves (3.17).

Case 1: $n < R$. Then $f = 0$ and hence $f'(n) = 0$. Because of (3.18) the IFT tells us that $y(n)$ is continuously differentiable in n. Consequently, (y, f) is continuously differentiable in n.

Case 2: $n > R$. Let $\tilde{J}$ denote the Jacobian of (3.16) with respect to (y, f):

$$\tilde{J} = \begin{pmatrix} \frac{\partial}{\partial y}\left(V_1(y, y, R)\right) & 0 \\ p_1(f, f, n, R)\Delta \frac{\partial}{\partial y}\left(V(y, y, R)\right) & \frac{\partial}{\partial f}\left(p_1(f, f, n, R)\right)\Delta V(y, y, R) - C''(f) \end{pmatrix}$$

But then

$$Det(\tilde{J}) = \frac{\partial}{\partial y}\left(V_1(y, y, R)\right)\left(\frac{\partial}{\partial f}\left(p_1(f, f, n, R)\right)\Delta V(y, y, R) - C''(f)\right) > 0$$

which by the IFT means that (y, f) is continuously differentiable in n.

Case 3: $n = R$. Because

$$y = \begin{cases} y(R) & R < n \\ y(n) & R \geq n \end{cases}$$

y is continuous in n but not differentiable at $n = R$. I now show that f is continuous in n also at $n = R$. We have

$$f = \begin{cases} f(n, R, \Delta) & R < n \\ 0 & R \geq n \end{cases}$$

Let

$$\psi(f) = p_1(f, f, n, R)\Delta V(y, y, R) - C'(f)$$

Then the Kuhn-Tucker necessary condition corresponding to the representative firm's opimization problem with respect to f is

$$\psi(f) + \lambda = 0 \qquad \lambda f = 0 \qquad f, \lambda \geq 0$$

Suppose $f > 0$. Then

$$\lim_{n \to R^+} \psi(f) = -C'(f) + \lambda$$

But then $f > 0$ implies $C'(f) > 0$ which implies $\lambda > 0$, a contradiction to optimality. Consequently, we must have $\lim_{n \to R^+} f(n, R, \Delta) = 0$. Hence f is continuous at $n = R$.

■

3.7.10 Proof of proposition 3.5

The Jacobian J of (3.16) is given by

$$J = \begin{pmatrix} V'' & 0 \\ p_1 \Delta V_2 & p'' \Delta V - C'' \end{pmatrix}$$

where $V'' \equiv \frac{\partial}{\partial y}(V_1(y, y, R))$ and $p'' \equiv \frac{\partial}{\partial f}(p_1(f, f, n, R))$. Hence $Det(J) = V''(p''\Delta V - C'') > 0$. Let $\psi \equiv p_1(f, f, n, R)\Delta V(y, y, R) - C'(f, \alpha)$. Then by Cramer's rule we have

$$sign(y'(\chi)) = sign\left(\frac{-V_{1\chi}}{V''}\right) = sign\,(V_{1\chi})$$

and

$$sign(f'(\chi)) = sign\left(\frac{p_1 \Delta V_2 V_{1\chi} - \psi_\chi V''}{Det(J)}\right) = sign\,(p_1 \Delta V_2 V_{1\chi} - \psi_\chi V'')$$

where $\chi \in \{R, n, \Delta, F, \alpha\}$. Then it is easy to see that $y'(R) < 0$, $f'(\Delta) > 0$ and $f'(\alpha) < 0$. If $\chi = R$ we get

$$sign(f'(R)) = sign(\underbrace{p_1 V_2 V_{13}}_{<0} - \underbrace{V''}_{<0}\underbrace{(p_{14} V + p_1 V_3)}_{?})$$

If $\chi = n$ we get $sign(f'(n)) = sign(p_{13})$. Hence the sign of $f'(R)$ and $f'(n)$ can only be determined under further assumptions.

■

3.7.11 Comparative statics under free entry: calculations

Assume parameter values such that an attention equilibrium ($n > R$) occurs endogenously. The Jacobian of (3.24) - (3.26) with respect to (y, f, n) is

$$J = \begin{pmatrix} V'' & 0 & 0 \\ p_1 \Delta V_2 & p'' \Delta V - C'' & p_{13} \Delta V \\ \frac{R}{n} \Delta V_2 & -p_1 \Delta V & -\frac{R}{n^2} \Delta V \end{pmatrix} \tag{3.59}$$

Then after some manipulation

$$sign\left(Det(J)\right) = sign\left(\frac{p''}{p_1} - p_{13}\frac{n^2}{R}\right)$$

But then (3.28) implies that $Det(J) < 0$. Let $\chi \in \{R, \Delta, F, \alpha\}$.

Proposition 3.9. *Suppose an attention equilibrium ($n > R$) occurs endogenously. Then the comparative statics are given by*

$$y'(\chi) = \tfrac{V_{1\chi}\Delta^2 V^2}{Det(J)}\left(\tfrac{R}{n^2}p'' - p_1 p_{13} - \tfrac{\omega}{n^2}\right)$$

$$f'(\chi) = \tfrac{\Delta V}{Det(J)}\left(V''\left(\tfrac{R\Pi_{f\chi}}{n^2} + \Pi_\chi p_{13}\right) - V_{1\chi}\Delta\tfrac{RV_2}{n}\left(p_{13} + \tfrac{p_1}{n}\right)\right)$$

$$n'(\chi) = \tfrac{\Delta V}{Det(J)}\left(\tfrac{V_{1\chi}\Delta V_2}{n}\left(np_1^2 + Rp'' - \omega\right) - \tfrac{V''}{R}\left(Rp''\Pi_\chi + p_1 R\Pi_{f\chi} - \Pi_\chi\omega\right)\right)$$

where $\Pi_\chi \equiv \frac{\partial}{\partial\chi}\left(\frac{R}{n}V(y,y,R) - F - C(f,\alpha)\right)$, $\Pi_{f\chi} \equiv \frac{\partial}{\partial\chi}\left(p_1(f,f,n,R)\Delta V(y,y,R) - C'(f,\alpha)\right)$, $\omega \equiv \frac{C''(f,\alpha)R}{\Delta V} \geq 0$ *and* J *is the Jacobian (3.59) with* $Det(J) < 0$.

Proof:

Follows immediately by applying Cramer's rule to

$$J\begin{pmatrix} y'(\chi) \\ f'(\chi) \\ n'(\chi) \end{pmatrix} = -\begin{pmatrix} V_{1\chi} \\ \Pi_{f\chi} \\ \Pi_\chi \end{pmatrix}$$

■

3.7.12 Proof of lemma 3.5

First note that under assumption 3.7 and by the definition of $V(y, \bar{y}, z)$ (see 3.2.3.1) we have a symmetric differentiable game that permits only interior equilibria and asserts the existence of a symmetric equilibrium (see 5.2 from chapter 5). Any symmetric equilibrium must satisfy $V_1(y^*, y^*, z) = 0$ with $y \in Int(S_y)$. Strategic complementarity excludes asymmetric equilibria (see theorem 5.2). Condition (3.18) corresponds to the condition that rules out multiple symmetric equilibria (see 5.3).

■

3.7.13 Proof of proposition 3.6

Let $y_1 = ... = y_n = y$. Then by the symmetry of V^j we have

$$\frac{\partial V^j(Y_A)}{\partial y_j} = \frac{\partial V^j(Y_{A'})}{\partial y_j} = V_1(y, y, R)$$

for all $A, A' \in B_j$. Hence

$$\sum_{A \in B_j} P_A \frac{\partial V^j(Y_A)}{\partial y_j} = V_1(y, y, R) \sum_{A \in B_j} P_A$$

But lemma 3.5 implies the existence of a unique y^* with $V_1(y^*, y^*, R) = 0$. The inexistence of multiple symmetric equilibria for any non-degenerate distribution of P_A follows from (3.40). Assumption 3.7 implies that no boundary equilibria exist. To proof the claim of uniqueness I apply the index theorem (see Vives (1999), p. 48), which requires the Jacobian of $-D_y$ to have a positive determinant whenever $D_y = 0$ holds. The j-th row of this Jacobian is

$$\left(-\sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_1} \quad \cdots \quad -\sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j^2} \quad \cdots \quad -\sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_n} \right)$$

I now show that (3.41) implies

$$-\sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j^2} > \sum_{g \neq j} \left| \sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g} \right| \tag{3.60}$$

The triangle inequality implies

$$\sum_{A \in B_j} P_A \sum_{g \neq j} \left| \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g} \right| = \sum_{g \neq j} \sum_{A \in B_j} P_A \left| \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g} \right| \geq \sum_{g \neq j} \left| \sum_{A \in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g} \right|$$

Symmetry of $V^j(\cdot)$ implies for any $A \in B_j$ that $\exists g(A) \neq j \in A$ such that

$$\sum_{g \neq j} \left| \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_g} \right| \leq (R-1) \left| \frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_{g(A)}} \right|$$

Hence a sufficient condition for (3.60) to hold is

$$-\sum_{A\in B_j} P_A \frac{\partial^2 V^j(Y_A)}{\partial y_j^2} > \sum_{A\in B_j} P_A (R-1)\left|\frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_{g(A)}}\right|$$

which is equivalent to

$$\sum_{A\in B_j} P_A \left(-\frac{\partial^2 V^j(Y_A)}{\partial y_j^2} - (R-1)\left|\frac{\partial^2 V^j(Y_A)}{\partial y_j \partial y_{g(A)}}\right|\right) > 0 \tag{3.61}$$

But (3.61) is obviously implied by condition (3.41). Symmetry further implies that whenever (3.60) holds then a similar statement holds for any other row of the $n \times n$ - Jacobian of $-D_y$. But then $-D_y$ must have a dominant diagonal whenever $D_y = 0$ and hence $-D_y$ must have a positive determinant[40] if $D_y = 0$. But then by the index theorem we conclude that the symmetric equilibrium must be unique.

■

3.7.14 Proof of proposition 3.7

Because (3.41) is required to hold the attention distribution does not affect the equilibrium price by proposition 3.6. The equilibrium price is determined vy $V_1(y, y, R) = 0$. Thus we only need to show that for the equilibrium price y the attention competition does not generate any asymmetric equilibria. Because $\Pi_{f_j f_j} = \sum_{A\in B_j} \frac{\partial P_A(\mathcal{F})}{\partial f_j^2} V(Y_A) - C''(f_j)$ and $\Pi_{f_j f_g} = \sum_{A\in B_j} \frac{\partial P_A(\mathcal{F})}{\partial f_j f_g} V(Y_A)$ condition (3.42) implies that $-\Pi_{f_j f_j} + \Pi_{f_j f_g} > 0$. But this means that the slope of player j's best resonse function with respect to f_g must be larger than -1:

$$\frac{\partial f_j(f_1, ..., f_n)}{\partial f_g} = -\frac{\Pi_{f_j f_g}}{\Pi_{f_j f_j}} > -1$$

But then there cannot be any asymmetric attention equilibria (see chapter 5). Condition (3.19) rules out the possibility of multiple symmetric equilibria. Consequently, there can only be one equilibrium in the price-attention game, the symmetric equilibrium.

■

[40] This follows because $Det(-D_y) = r_1 \cdot ... \cdot r_n$ and all eigenvalues r_k must have positive real parts.

3.7.15 Proof of corollary 3.3

For simplicity I set $j = 1$ and $g = 2$. The following argument obviously remains the same if any different players were used.

I use the following decomposition of B_1. Let $\hat{B}_2 \equiv \{A \in B_1 : 1 \in A \wedge 2 \in A\}$ and $\hat{B}_{-2} \equiv \{A \in B_1 : 1 \in A \wedge 2 \notin A\}$. Hence

$$\Pi^1 = \sum_{A \in \hat{B}_1} P_A V(Y_A) + \sum_{A \in \hat{B}_2} P_A V(Y_A) - C(f_1) - F$$

Thus according to proposition 3.7 we need to proof

$$-\frac{\partial^2 P_A}{\partial f_1^2} + \frac{\partial^2 P_A}{\partial f_1 \partial f_2} > 0 \tag{3.62}$$

for $f_1, ..., f_n \in (0, \infty)$ and any $A \in B_1$. Suppose $A \in \hat{B}_{-2}$. Then P_A takes on the form

$$P_A = \frac{f_1 f_h}{\sum f_j} \left(\frac{1}{\sum f_j - f_1} + \frac{1}{\sum f_j - f_h} \right)$$

where $3 \leq h \leq n$. Now suppose $A' \in \hat{B}_2$. Then

$$P_{A'} = \frac{f_1 f_2}{\sum f_j} \left(\frac{1}{\sum f_j - f_1} + \frac{1}{\sum f_j - f_2} \right)$$

In fact with $R = 2$ there can only be one such set: $A' = \{1, 2\}$. Differentiation yields:

$$-\frac{\partial^2}{\partial f_1^2} P_A + \frac{\partial^2}{\partial f_1 \partial f_2} P_A = \frac{1}{\left(\sum f_j - f_h\right)^2} - \frac{1}{\left(\sum f_j\right)^2} > 0$$

and

$$-\frac{\partial^2}{\partial f_1^2} P_{A'} + \frac{\partial^2}{\partial f_1 \partial f_2} P_{A'} = \frac{2\left(\sum f_j - f_1 - f_2\right)}{\left(\sum f_j - f_2\right)^3} > 0$$

Consequently, (3.62) is satisfied for any $A \in B_1$.

■

3.7.16 Proof of corollary 3.4

For $C(f) = \theta f^{\eta}$ we have

$$C''(f) = \frac{(\eta-1)}{f} C'(f) \tag{3.63}$$

Using $V_2 = 0$, (3.63) and (3.16) in (3.44) gives

$$\frac{|p_{12}(f, f, R, n)|}{-\left(p_{11}(f, f, R, n) - \frac{(\eta-1)}{f} p_1(f, f, n, R)\right)} < 1 \tag{3.64}$$

Further calculate

$$p_{12}(f, f, R, n) = G(f) \left(\left(\sum_{i=1}^{R} \frac{1}{f(1+n-i)} \right)^2 - \sum_{i=1}^{R} \frac{n-i}{f^2(1+n-i)^2} \right) \tag{3.65}$$

and from (3.54)

$$p_{11}(f, f, R, n) = -G(f) \left(\left(\sum_{i=1}^{R} \frac{1}{f(1+n-i)} \right)^2 + \sum_{i=1}^{R} \frac{1}{f^2(1+n-i)^2} \right) \tag{3.66}$$

With (3.20) and $V_2 = 0$ condition (3.44) is equivalent to

$$\left| \underbrace{\left(\left(\sum_{i=1}^{R} \frac{1}{1+n-i} \right)^2 - \sum_{i=1}^{R} \frac{n-i}{(1+n-i)^2} \right)}_{=K} \right| < \left(\left(\sum_{i=1}^{R} \frac{1}{(1+n-i)} \right)^2 + \sum_{i=1}^{R} \frac{1}{(1+n-i)^2} \right) + (\eta-1) \sum_{i=1}^{R} \frac{1}{(1+n-i)} \tag{3.67}$$

which obviously is rather satisfied for larger η. Thus I set $\eta = 2$. If $K \geq 0$ then (3.67) obviously holds. If $K < 0$ then (3.67) also holds because

$$\begin{aligned}
0 &< 2\left(\sum_{i=1}^{R} \frac{1}{(1+n-i)} \right)^2 + \sum_{i=1}^{R} \frac{1-n+i}{(1+n-i)^2} + \sum_{i=1}^{R} \frac{1}{(1+n-i)} \\
&= 2\left(\sum_{i=1}^{R} \frac{1}{(1+n-i)} \right)^2 + \sum_{i=1}^{R} \frac{1-n+i}{(1+n-i)^2} + \sum_{i=1}^{R} \frac{1+n-i}{(1+n-i)^2} \\
&= 2\left(\sum_{i=1}^{R} \frac{1}{(1+n-i)} \right)^2 + \sum_{i=1}^{R} \frac{2}{(1+n-i)^2}
\end{aligned}$$

∎

4

Applications and Extensions

In the first section of this chapter I introduce attention competition from chapter 3 to the Salop model of circular product differentiation. Given their location on the unit circle the firms simultaneously choose their price and their attention effort. I show that if we use the ACF to determine the attention probabilities then the Salop model generates a symmetric opponent form that is similar to the general symmetric opponent form of chapter 3. Hence we can use the framework of chapter 3 to discuss existence and the comparative statics of the equilibrium in the Salop model under limited attention. In this model the number of active firms, n, has the nice interpretation of a measure of overall diversity. What we additionally gain by the Salop model, compared to chapter 3, is the possibility to dicuss welfare implications of limited attention. I show that under limited attention the usual negative relationship between overall diversity n and average consumer transportation costs is inverted. This implies that, under limited attention, average consumer utility *decreases* in n whereas under unlimited attention the opposite holds. This explains why a planning authorities' primary object is to cut back diversity under limited attention.

In section 4.2 I combine the model of informative advertising from chapter 2 with attention competition from chapter 3. This means that, other than in the Internet economy from chapter 3, the information sets I_i of the consumers are heterogeneous and depend on the information effort of the advertising firms. I show that limited attention implies higher average transportation costs (a result from chapter 4.1) also in the case where firms can choose the fraction of consumers they want to inform. Comparing the model with the conventional model of Grossman and Shapiro I provide some evidence that my model may give a better fit to the development of the advertising-consumption shares as reported in the introduction.

In the sections 4.3 and 4.4 I extend the theory of chapter 3. In section 4.3 I discuss the implications of heterogeneity in the attention threshold R on market equilibrium and consumer welfare in case of the Salop model. In section 4.4 I illustrate that unilateral advantages in attracting attention of a firm can be instrumentalised by the firm to overcome other technological inefficiencies. This may lead to a crowding-out of more efficient firms, as measured by production technology, in the free-entry equilibrium if these firms fail to get sufficient attention.

4.1 Attention competition in the Salop model

In this section I introduce attention competition, as developed in chapter 3, to the circular model of ideal variety from chapter 2. The main difference to chapter 2 is, that I consider the case of the Internet economy (see 3.2.1). Whereas in the model of chapter 2 the reach of a firm, ϕ_j, was endogenously determined, I now impose $\phi_j = 1$ for any firm that pays the fixed cost $F > 0$. This means that the consumers receive information of all active firms. However, perception is limited by R. If $R \geq n$ (consumers perceive all existing varieties) then the model of this chapter coincides exactly with the conventional Salop model (Salop (1979)). Consumers have the same utility[1] as in chapter one: $u_i(j) = v - tw_i(j) - y_j$. I now introduce limited attention to the model. For simplicity, I normalise population size to one. As we will see, the Salop model generates a symmetric opponent form of the profit function that matches the general specification of the last chapter.

[1]Unit demand makes the model as simple as possible. The basic results do not depend on the assumption of unit demand as can be shown by solving the model e.g. with a Wong-type utility function $u_i = \sum_{j \in A_i} \frac{q_{ji}}{1+tw_i}$ where q_{ji} is demand of consumer i for the commodity of firm j (see Wong (1995), p. 260). Intuitively, this is the case because different specifications for u_i only affect N_k but not the way how limited attention enters the profit function of the firm.

4.1.1 Demand and profit for $n > R$

As in chapter 2 for a given price y of the representative firm and given $\bar{y}$ I partition the set of consumers into n groups, $k = 1, ..., n$ where the k-th group is the subset of consumers to whom the representative firm would offer the k-th highest net utility of the n firms under unlimited attention. Under unlimited attention (if $R \geq n$) consumers compare all n varieties which induces a strong price competition and in equilibrium firms can only sell to their prime segment $k = 1$ and consumers purchase from their "closest" location (in the sense of next to ideal variety). With limited attention consumers consider only a subset of all available commodities. Hence from the viewpoint of the representative firm it may be possible to make a sale to consumers far away if the firm is perceived but a better firm (in terms of net utility) is not perceived. With this partitioning of the set of all consumers into n favourite groups the expected demand function of the representative firm is

$$E[Q, R] = \sum_{k=1}^{n} E\left[q_k, R\right] N_k \tag{4.1}$$

where $E\left[q_k, R\right] \in [0, 1]$ denotes expected demand from a member of group k and N_k is the number of consumers in group k. Then by the law of total expectations we have

$$E\left[q_k, R\right] = p(A)p\left(k|\, A\right) \tag{4.2}$$

where $p(A)$ denotes the probability of the representative firm (firm j) to be in the attention set[2] of a member of group k and $p\left(k|\, A\right)$ is the conditional probability of offering the highest net utility among all other perceived firms. I set $y_j = y$ and $f_j = f$ as well as $y_g = \bar{y}$ and $f_g = \bar{f}$ for all $g \neq j$. By chapter 3 we then may set $p(A) = p\left(f, \bar{f}, n, R\right)$, where $p\left(f, \bar{f}, n, R\right)$ is the probability of being perceived (i.e. of being in an attention set). Throughout the rest of chapter 4.1 I assume that $p(f, \bar{f}, n, R)$ is given by the ACF. Then

$$p(A) = p\left(f, \bar{f}, n, R\right) = 1 - \prod_{i=1}^{R} \left(1 - \frac{f}{f + (n-i)\bar{f}}\right) \tag{4.3}$$

[2] I always assume this probability to be independent of group identity k. This means that firms cannot identify consumers and hence cannot direct their messages to a certain subset of consumers. Therefore $p(A)$ is the same for all consumers and independent of group identity k.

Moreover, we have

$$p\left(k\,|A\right)=\prod_{i=1}^{R-1}\left(1-\frac{(k-1)\bar{f}}{(n-i)\bar{f}}\right)=\prod_{i=1}^{R-1}\left(\frac{1+n-k-i}{n-i}\right) \tag{4.4}$$

This is the conditional probability that given the representative firm made it into the attention set none of the $k-1$ superior[3] firms makes it into the attention set. Thus

$$E\left[q_k,R\right]=\left(1-\prod_{i=1}^{R}\left(1-\frac{f}{f+(n-i)\bar{f}}\right)\right)\prod_{i=1}^{R-1}\left(\frac{1+n-k-i}{n-i}\right) \tag{4.5}$$

Assuming $c=0$ and $n>R$ the profit function of the representative firm is

$$\begin{aligned}\Pi &= yE\left[Q,R\right]-F-C(f)\\ &= yp(f,\bar{f},n,R)\sum_{k=1}^{n}p\left(k|\,A\right)N_k-F-C(f)\end{aligned}$$

The group measures N_k are the same as in chapter one:

$$N_1=\frac{\bar{y}-y}{t}+\frac{1}{n}\qquad N_k=\frac{1}{n}\quad 2\leq k\leq n-1\qquad N_n=\frac{1}{n}-\frac{\bar{y}-y}{t} \tag{4.6}$$

4.1.2 Equilibrium

In this section I derive the symmetric equilibrium of the Salop model with attention competition. First, I concentrate on the case where n is given exogenously and determine (y,f) in the equilibrium. Then I move on to the free-entry equilibrium that determines (y,f,n). Finally, I relate the comparative-statics of the Salop model with attention competetion to the general results of chapter 3.4.

If $R\geq n$ then $E[Q,n]=N_1$. As all n varieties are perceived it is only possible to make a sale to the prime segment $k=1$. The model then corresponds to the conventional

[3] As in chapter 2 all reference to ”superior” and ”inferior” are in terms of net utility and relative to the representative firm.

Salop model (Salop (1979)). Thus we have

$$\Pi(y,f) = \begin{cases} \underbrace{y\, p\left(f,\bar{f},n,R\right) \sum_{k=1}^{n} p\left(k\,|A\right) N_k}_{=E[Q,R]} - F - C(f) & R < n \\ yN_1 - F & R \geq n \end{cases} \tag{4.7}$$

In the notation of the last chapter we have

$$V\left(y,\bar{y},\min\{R,n\}\right) = \begin{cases} y \sum\limits_{k=1}^{n} p(k\,|A) N_k & R < n \\ yN_1 & R \geq n \end{cases}$$

Hence from (4.7) we see that the Salop-model with limited attention has a symmetric opponent form similar to (3.9) from chapter 3.

Lemma 4.1. *For $\bar{y} = y \leq v - t/2$, $\bar{f} = f > 0$ and $z = min\{R,n\} \geq 2$ we have*

a) If $n > R$ we have $p\,(1\,|A) = 1$ and $p\,(n\,|A) = 0$

b) $V(y,y,z) = y/z$

c) $E[Q,z] = 1/n$

Proof: Appendix B (4.6.1)

Proposition 4.1. *Suppose $v \geq t$ and $R, n \geq 2$ and $p(f,\bar{f},n,R)$ is represented by the ACF. Then a single symmetric equilibrium (y,f) exists*

i) if $n > R$ then

$$y = t/R \tag{4.8}$$

$$p_1(f,f,n,R)\frac{t}{R^2} = C'(f) \tag{4.9}$$

$$\Pi = \frac{t}{Rn} - F - C(f) \tag{4.10}$$

and $f \in (0,\infty)$

ii) if $n \leq R$ then $f = 0$ and

$$y = t/n \tag{4.11}$$

$$\Pi = \frac{t}{n^2} - F \tag{4.12}$$

Proof: Appendix B (4.6.2)

I now turn to the case where n is determined endogenously by the zero-profit condition.

Proposition 4.2. *Suppose $p(f, \bar{f}, n, R)$ is represented by the ACF and $R \geq 2$. If $t \geq 4F$ the free entry game has a single SPE with $2 \leq n < \infty$. If $\sqrt{\frac{t}{F}} > R$ then (y, f, n) is approximately determined by*

$$\begin{array}{l} y = t/R \\ \frac{n-R}{n^2 f}\frac{t}{R} = C'(f) \\ \frac{t}{Rn} = F + C(f) \end{array} \tag{4.13}$$

If $\sqrt{\frac{t}{F}} \leq R$ then

$$\begin{array}{l} y = \sqrt{tF} \\ f = 0 \\ n = \sqrt{\frac{t}{F}} \end{array} \tag{4.14}$$

Proof: Appendix B (4.6.3)

4.1.2.1 Comparative statics under free entry

In this section I assume that $R < \sqrt{t/F}$ and use $C(f) = \theta f^{\eta}$. Because with this example we have $V(y, y, R) = y/R$ the comparative static results from chapter 3.4.2.2 apply. Hence we unambiguously have $f'(R) < 0$ for this example. Using lemma 4.1 we get $\varepsilon_v = -1$ because $\tilde{v}(R) = \tilde{V}(y(R)) = y(R) = t/R$. Hence $sign\,(n'(R)) = sign\,(R + \varepsilon_v\,(R + n(\eta - 1))) = sign\,(-n(\eta - 1))$ and $n'(R) \leq 0$ with $n'(R) = 0$ if and only if $\eta = 1$. Consequently, the Salop model predicts the number of active firms and hence diversity to increase if consumers become more inattentive (R decreases) as the revenue effect dominates. This occurs because in this models markups are inversely proportional to the number of perceived firms (see (4.8) and (4.11))[4]. This is a different result than in the ACF-CES example where both cases $n'(R) > 0$ and $n'(R) < 0$ are

[4]The result that $n'(R) \leq 0$ for $\eta \geq 1$ is also obtained if the Wong distance function is used with the assumption of unit demand. However, if we allow for continuous demand then the condition for $n'(R) \leq 0$ depends on η. Intuitively, this is the case because with elastic demand consumers can evade some price competition which leads to a smaller value of ε_v.

possible (see chapter 3.4.2.2). Because of the simple structure of the equilibrium equations it is possible to analytically investigate how limited attention affects the magnitude of certain changes. Let $n^{NL} = \sqrt{t/F}$. We can use (3.34) from the last chapter to find $\frac{t}{Rn^2\eta}(n(\eta-1)+R) = F$ which can be solved for

$$n^L = \frac{t(\eta-1)+\sqrt{t^2(\eta-1)^2+4tFR^2\eta}}{2FR\eta} \tag{4.15}$$

Comparing n^L and n^{NL} we see that a higher t increases equilibrium diversity in both cases. Intuitively, as t increases the varieties become weaker substitutes which implies that a higher equilibrium price can be sustained which, because of $V_2(y,y,R) = v'(y) = 1 > 0$, translates into a higher equilibrium revenue. Define $\varepsilon_t \equiv \frac{n'(t)t}{n(t)}$ as well as $\varepsilon_F \equiv \frac{n'(F)F}{n(F)}$. Then we get

$$\varepsilon_t^{NL} = 1/2 \qquad \varepsilon_t^L = \frac{1}{2} + \frac{\sqrt{t}(\eta-1)}{2\sqrt{t(\eta-1)^2+4FR^2\eta}}$$

Hence $\varepsilon_t^{NL} < \varepsilon_t^L$ for $\eta > 1$. This means that the magnitude of change of $n(t)$ is larger under limited attention. Intuitively, this occurs because with unlimited attention equilibrium prices depend negatively on n. Then the additional firms that enter the market reinforce price competition and reduce markups which derates the initial gain and reduces entry. In contrast, under limited attention equilibrium prices do not depend on n and hence the deration effect is absent. Thus n reacts stronger to a change of t in an attention economy. A similar argument holds for a change in F. We have

$$\left|\varepsilon_F^{NL}\right| = 1/2 \qquad \left|\varepsilon_F^L\right| = \frac{1}{2} + \frac{\sqrt{t}(\eta-1)}{2\sqrt{t(\eta-1)^2+4FR^2\eta}}$$

Hence $\left|\varepsilon_F^{NL}\right| < \left|\varepsilon_F^L\right|$ for $\eta > 1$. Thus a decrease of fixed costs means that the induced rate of entry is higher under limited attention. The reason again is that with limited attention equilibrium prices are independent of the actual market size n. Figure 4.1 illustrates these results. In the figure we see that the difference between the curves increases as t increases (F decreases) which reflects the magnitude effect of limited attention.

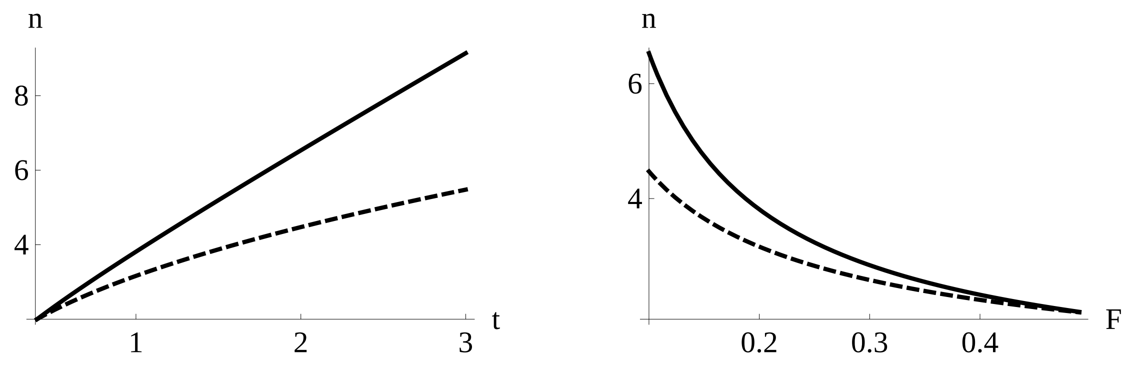

Figure 4.1: F, t - effects under limited attention (solid) and unlimited attention

4.1.3 Welfare

In this section I investigate the implications of limited attention on welfare.

As Grossman and Shapiro show the average distance travelled by a consumer who purchases his k-th favourite variety (for equal prices) is given by $\bar{w}_k = (2k-1)/4n$ (Grossman and Shapiro (1984) p. 74). For $n > R$ the probability that a consumer purchases at his k-th best firm is $E[q_k, R]$.[5] If $R \geq n$ then $E[q_1, n] = 1$ and $E[q_k, n] = 0$ for $k > 1$. The expected distance a consumer travels is given by

$$\bar{w} = \begin{cases} \sum_{k=1}^{n} E[q_k, R]\bar{w}_k & R < n \\ \frac{1}{4n} & R \geq n \end{cases} \tag{4.16}$$

Proposition 4.3. *In the symmetric equilibrium ($\bar{y} = y$ and $\bar{f} = f$) average transportation distance is:*

i) if $n > R$:

$$\bar{w}^L = \frac{(n+1)}{2n(R+1)} - \frac{1}{4n} \tag{4.17}$$

ii) if $R \leq n$: $\bar{w}^{NL} = 1/4n$.

Hence $\bar{w}^L > \bar{w}^{NL}$, $\frac{\partial \bar{w}^{NL}}{\partial n} < 0$ and $\frac{\partial \bar{w}^L}{\partial n} > 0$.

Proof: Appendix B (4.6.4)

[5] The consumer purchases his k-th best variety if the other $R-1$ perceived varieties are inferior. There are $n-k$ inferior firms. But if $n-k < R-1$ then at least one perceived firm must be superior. Hence nobody consumes worse than the $(1+n-R)$th variety.

We can write average transportation distance as

$$\bar{w}(n) = \begin{cases} \frac{n+1}{2n(R+1)} - \frac{1}{4n} & R < n \\ \frac{1}{4n} & R \geq n \end{cases} \tag{4.18}$$

We see from (4.18) that $\bar{w}(n)$ is continuous in n and $\lim_{n\to\infty} \bar{w}(n) = \frac{1}{2(R+1)} > 0$ and $\bar{w}''(n) = -\frac{R-1}{2n^3(R+1)} < 0$ if $n > R$. Hence the positive effect of n on average transportation distance is strongest if $n \approx R$ and diminishes as n gets larger. This is illustrated in the next figure. Under unlimited attention the relationship between average transportation distance and

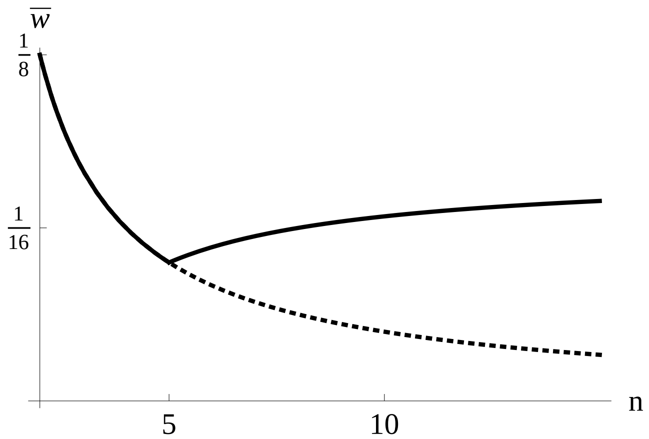

Figure 4.2: The function $\bar{w}(n)$ (solid) for $R = 5$

diversity is negative, but positive under limited attention. Why is this the case? The answer to the question is not trivial. Let $n = 5$ and consider the k-th best variety for a fixed consumer. If we add another firm to the circle and, as usually, assume that the firms reallocate symmetrically around the circle then our fixed consumer still has $k-1$ superior varieties but $6-k > 5-k$ inferior varieties. What this means is that for any given k if we increase n then the consumer has a higher chance of picking an inferior variety relative to k. On the other side a higher n means that firms move closer together, i.e. the distance between two firms declines in n. This means that choosing an inferior variety (relative to k) may lead to a less severe punishment in terms of additional transportation costs if n is higher. Moreover, the consumer may benefit more by choosing a superior variety if n is higher because then firms are closer together. The calculations of proposition 4.3 show the choice probability effect (the chance of choosing an inferior variety) to dominate the effect of the firms moving closer together. Hence in the Salop model with limited attention the "confusion" of consumers by more information (more diversity) implies that consumer on average choose a worse variety (in the sense that average transportation distance is larger).[6]

This result is of considerable importance. Let $T = t\bar{w}$ denote average transportation costs of a consumer. Without limited attention the model predicts that if equilibrium diversity increases (n increases) then the average consumer is better off because i) all consumers pay a lower price and ii) the average consumer purchases a variety closer to his ideal variety. Proposition 4.3 shows that under limited attention things are completely different. The average consumer is *worse* off if diversity increases because i) prices do not react to an increase of n but ii) the average consumer purchases a variety further away from his ideal variety.

It has been argued that the introduction of the internet has reduced the fixed costs F (see the discussion on page 39). In the conventional Salop equilibrium lower F, ceteris paribus, means higher diversity n and lower prices (see (4.14)). But a lower F also increases the chance that an attention equilibrium occurs. If this is the case a further reduction of F has no effect on prices but *increases* average transportation costs T. This is illustrated in figure 4.3. In the upper picture of figure 4.3 we see that under limited attention a further reduction of F increases average transportation costs. This were not the case under unlimited attention as the dashed line suggests. Limited attention then implies average consumer utility to decrease in F because average transportation costs increase and prices remain constant. This is depicted in the lower picture.

4.1.3.1 Socially optimal level of diversity

We know from the canonical Salop model that the market overprovides diversity. This occurs because firms do not take into account their aggregate effect on transportation costs whereas the planer weights the marginal cost of an additional firm (which is F) against its social value (which is the average reduction in transportation costs). I now show that this result extends to the case of limited attention. Moreover, I show that the discrepancy between the market solution and the social optimal value of diversity is enlarged under attention competition the more inelastic attention competition becomes.

[6] In Appendix A I investigate the result more thoroughly.

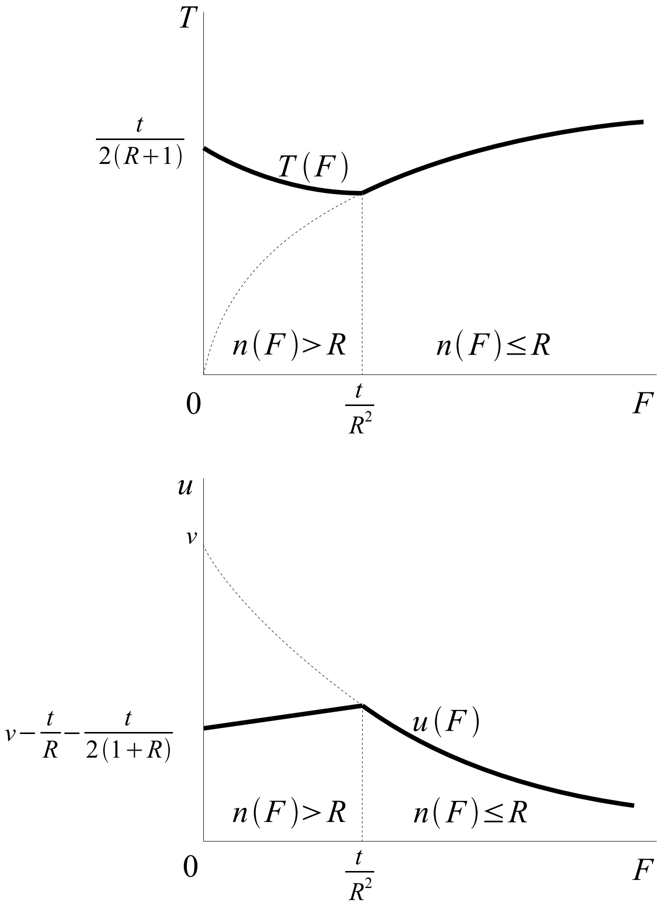

Figure 4.3: Transportation costs and average consumer utility as a function of F

The welfare function is $W = v - nF - T(n) - nC(f)$ with $T(n) = t\bar{w}(n)$ where $T(n)$ are average transportation costs of consumers. The planer chooses (f, n) in order to opimize the welfare function W. Hence

$$T(n) = \begin{cases} \frac{t(n+1)}{2n(R+1)} - \frac{t}{4n} & R < n \\ \frac{t}{4n} & R \geq n \end{cases} \tag{4.19}$$

Proposition 4.4. *Suppose $R < \sqrt{\frac{t}{F}}$. Then the optimal level of diversity n^P is given by*

$$n^P = \begin{cases} \frac{1}{2}\sqrt{\frac{t}{F}} & \sqrt{\frac{t}{F}} \leq 2R \\ R & \sqrt{\frac{t}{F}} > 2R \end{cases} \tag{4.20}$$

Moreover, the planer always chooses $f^P = 0$.

Proof: Appendix B (4.6.5)

The planer always chooses $f^P = 0$ because any other level of f is costly but does not lead to a better match between firms and consumers.[7] A similar result is obtained by Falkinger (Falkinger (2008)). If we have $R < \sqrt{\frac{t}{F}}$ then by proposition 4.2 an attention equilibrium occurs, i.e. we have $n > R$. From (4.20) we see that $n^P \leq R$. Thus the market overprovides diversity also under limited attention.

The planer problem is illustrated in figure 4.4. In the figure the dashed line corresponds to the welfare function under unlimited attention and the solid line corresponds to the true welfare function of the planer. We never have $n^P > R$ simply because the only

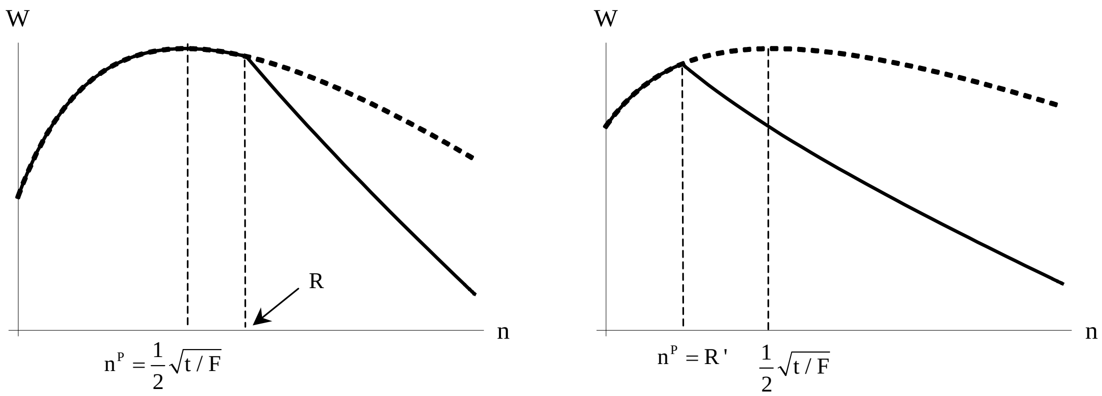

Figure 4.4: Planer solution $n^P(R)$

socially relevant reason to increase diversity is that transportation costs decrease due to a better match of consumer to variety. Under limited attention this effect does no longer exist as attention constrained consumers cannot benefit from the increased diversity. If the size of the attention set is sufficiently large then the planer chooses $n^P < R$, the same level of diversity that he had chosen in the case of unrestricted attention (left picture). If however R is small then choosing $n^P < R$ may be socially too expensive (right picture) because this implies very high transportation costs. Hence the best choice of the planer then simply is $n^P = R$.

Moreover, limited attention may sincerely increase the market inefficiency. Let $\rho \equiv n - n^P$ denote excessive diversity.

[7]This follows because, as the firms, the planer cannot discriminate between the consumers.

Proposition 4.5. *Under limited attention $n^P \leq R < n$. If consumers are less attentive this implies higher excessive diversity as $\rho'(R) \leq 0$. If $\eta > 1$ then $\rho'(R) < 0$. Moreover, excessive diversity increases in η.*

Proof: Appendix B (4.6.6)

Suppose we have $\eta > 1$. The intuition behind proposition 4.5 is that if R decreases (consumers are less attentive) then because $n'(R) < 0$ market diversity increases. At the same time we know from proposition 4.4 that the planer never chooses $n^P > R$. As n increases and n^P does not increase (but may decrease) excessive diversity increases if consumers become less attentive. Moreover, excessive diversity increases with η because in the Salop model we must have $n'(\eta) > 0$ (see the discussion on page 97 of chapter (3.36)) but n^P does not depend on η because $f^P = 0$. Taken together these facts imply that if $\sqrt{\frac{t}{F}} > 2R$ (such that $n^P = R$) then the larger the difference between n and R is, the larger is the welfare loss due to limited attention in the economy.[8]

4.2 Attention competition and informative advertising

In this section I introduce attention competition into the model of informative advertising from chapter 2 based on the approach of chapter 3. The model in this section basically merges the model of chapter 2 with the model of section 4.1 of this chapter. Every firm j simultaneously and non-cooperatively chooses its strategy, the vector (y_j, ϕ_j, f_j), where ϕ_j corresponds to the fraction of consumers firm j wants to inform. As before, I restrict myself to the case of symmetric firms and only discuss symmetric equilibria. Formally, we have to deal with a three-dimensional static symmetric n-player game.

In section 4.2.1 I derive the symmetric opponent form of this game and present the equilibrium equations in 4.2.2. I illustrate that the model of limited attention and informative advertising fits the US-data on advertising and consumption expenditures better

[8] This also is a conclusion in Falkinger's model of limited attention (Falkinger (2008)). However, in my model the number of firms (overall diversity) always exceeds the socially optimal level which is not the case in his model. A further difference is that in my model the planer might eventually choose $n^P < R$ whereas in his model optimal perceived diversity always corresponds to τ_0.

than the conventional model of informative advertising and unlimited attention in section 4.2. Finally, I show in section 4.2.4 that the result from section 4.1.3 on average transportation costs (limited attention implies higher transportation costs) extends to the model of informative advertising.

4.2.1 The symmetric opponent form

Every consumer i is attributed an information set I_i which comprises all the commodities the consumer is informed of. A completely uninformed consumers has $I_i = \emptyset$. If the consumer is informed e.g. of commodity j then $j \in I_i$. As in the previous chapters limited attention means that a maximum of $1 < R < \infty$ items of the information set I_i are perceived. In chapter 2 information sets I_i can be different among consumers as I_i depends on advertising and is endogenously determined. I denote by $\tilde{A}_i$ the attention set of consumer i. $\tilde{A}_i$ is the set of varieties the consumer effectively considers in making his decision. We must have $\tilde{A}_i \subset I_i$. If $|I_i| \leq R$ we have $\tilde{A}_i = I_i$, but $|I_i| > R$ implies $\tilde{A}_i \subsetneq I_i$. In case of an attention constrained consumer competition for this consumer's attention emerges if the firm can by some means influence the chance of being perceived, i.e. the chance of being contained in the attention set of the consumer. In the model of this section a firm has two strategic information variables it can choose. A firm decides i) what fraction of the population to inform (ϕ) and ii) how salient its messages should be (f). To illustrate the difference between these two information instruments suppose a firm with a fixed advertising budget plans a newspaper advertising campaign. Then this firm faces a trade-off between in how many newspapers to place an ad versus the placement or size of the ad in a given newspaper. I take firm j as the representative firm and set $y_j = y$, $\phi_j = \phi$ and $f_j = f$ as well as $y_g = \bar{y}$, $\phi_g = \bar{\phi}$ and $f_g = \bar{f}$ for any $g \neq j$. The remainder of this section is devoted to deriving the symmetric opponent form of the symmetric game.

In chapter 2.2.1.2 I showed that the representative firm's expected demand from a member of favourity group k given that this member received information from the representative firm[9] has the form (see (2.7))

$$E\left[q_k, R\,|j \in I_k\right] = B_1 + B_2$$

In this expression B_1 corresponds to the firm's conditional demand if the consumer has limited information (if he receives ads of less than R opponents). B_2 corresponds to the firm's conditional demand if the consumer has limited attention (if he receives ads of at least R opponents). If a consumer receives an ad from the representative firm and less than R ads from the opponents of the representative firm then this consumer considers all ads he receives. Hence the relative salience of the messages plays no role for perception. This means that $B_1 = B_1(\bar{\phi}, n, R)$ as determined in (2.7). Limited attention, and the competition for attention, only matters for expected demand if the consumer receives ads of at least R opponents. In such a case the consumer is not capable of considering his entire information set and the relative salience of the messages determines the probability with which an item is perceived. Hence we have $B_2 = B_2(\bar{\phi}, n, R, f, \bar{f})$. In the appendix (see 4.6.7) I show that, under the ACF, the conditional probability to make a sale to a member of group k who received ads of exactly $z \geq R$ opponents is

$$P_k(S(z)\,|z \geq R, j \in I_k) =$$

$$\sum_{s=0}^{k-1} \underbrace{\bar{\phi}^z(1-\bar{\phi})^{n-1-z} \begin{pmatrix} k-1 \\ s \end{pmatrix} \begin{pmatrix} n-k \\ z-s \end{pmatrix}}_{G_1} \underbrace{\overbrace{p\left(f, \bar{f}, 1+z, R\right)}^{p(A)} \overbrace{\prod_{i=1}^{R-1} \frac{1+z-i-s}{(1+z-i)}}^{p(k|A)}}_{G_2} \tag{4.21}$$

Suppose a member of group k receives information of $z \geq R$ opponents. Then G_1 is the probability to be in an information set with s superior firms and $z-s$ inferior firms[10]. G_2 is the probability to be the best perceived firm given an information set with z opponents and s superior firms. This probability depends positively on f, as the chance of perception, $p(A)$, depends positively on f. By a similar argument the probability depends negatively

[9]This means that $j \in I_k$. E.g. the consumer buys a newspaper in which the firm has acquired an advertising slot.

[10]Note that for $\bar{\phi} > 0$ we have $G_1 > 0$ only if $s \leq z$.

on $\bar{f}$. Expected conditional demand from a member of group k then is

$$E\left[q_k, R\,|j \in I_k\right] = B_1(\bar{\phi}, n, R) + B_2(\phi, n, \bar{R}, f, \bar{f}) \tag{4.22}$$

where

$$B_1(\bar{\phi}, n, R) = \sum_{z=0}^{R-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \begin{pmatrix} n-k \\ z \end{pmatrix}$$

and $B_2(\phi, n, \bar{R}, f, \bar{f})$ is given by

$$B_2(\bar{\phi}, n, R, f, \bar{f}) = \sum_{z=R}^{n-1} P\left(S(z)\,|z \geq R, j \in I_k\right)$$

Suppose advertising costs are given by $\theta r \Delta m + m\gamma C(f)$. This means that the cost of acquiring m channels depends on the base cost $\theta r \Delta$ of a channel but also on the cost of attention effort per channel. Intuitively, this captures the idea that a firm can buy a better "slot" in a given channel, e.g. by purchasing a larger space in a newspaper to place its ad in, or by acquiring a position on the front page. Then, setting $c = 0$ and defining $A(\phi) = \theta r \Delta m(\phi)$ where $\phi(m)$ is given by (2.5) (see chapter 2.1.2), the symmetric opponent form of the profit function is (see (2.8))

$$\begin{aligned} \Pi(y, \phi, f) &= \phi y \Delta \left(\left(\tfrac{\bar{y}-y}{t} + \tfrac{1}{n}\right) E\left[q_1, R\,|j \in I_1\right] + \tfrac{1}{n} \sum_{k=2}^{n-1} E\left[q_k, R\,|j \in I_k\right] \right) \\ &\quad + \phi y \Delta \left(\tfrac{1}{n} - \tfrac{\bar{y}-y}{t}\right) E\left[q_n, R\,|j \in I_n\right] - F - A(\phi)\left(1 + \kappa C(f)\right) \end{aligned} \tag{4.23}$$

where $\kappa \equiv \gamma/(\theta r \Delta)$.

4.2.2 The symmetric equilibrium

The parameters of the game are $\{\Delta, F, n, R, t, \theta, r, a\}$. In a symmetric equilibrium all active firms choose the same strategy, the vector (y, ϕ, f). As in chapter 2 I say that an attention equilibrium occurs if $n\phi > R$, where $n\phi$ corresponds to the average size of an information set[11]. If $n\phi \leq R$, i.e. a conventional equilibrium occurs, then $f = 0$ and y and ϕ are determined by (2.21) and (2.22).

Assuming the existence of a symmetric equilibrium with $n\phi > R$ the three equilib-

[11]That is, on average a consumer is informed of $n\phi$ different varieties.

rium equations for $(1-\phi)^n \approx 0$ can be derived from the first-order conditions of (4.23), evaluated at $\bar{y} = y$, $\bar{f} = f$ and $\bar{\phi} = \phi$:

$$\Pi_y = 0 \quad \Rightarrow \quad y = \frac{t}{R} \tag{4.24}$$

$$\Pi_\phi = 0 \quad \Rightarrow \quad \frac{y\Delta}{n\phi} = A'(\phi)\,(1 + \kappa C(f)) \tag{4.25}$$

$$\Pi_f = 0 \quad \Rightarrow \quad \frac{\phi y \Delta}{n} \sum_{k=1}^{n} \frac{\partial E\left[q_k, R\,|j \in I_k\right]}{\partial f} - A(\phi)\kappa C'(f) = 0 \tag{4.26}$$

It should be clear that equation (4.24) must corresponds exactly to (2.21) (for $c = 0$ and $R < n\phi$). Moreover, equation (4.25) must correspond to (2.22) (for $c = 0$) up to the factor $(1 + \kappa C(f)$. This factor appears as the marginal cost of extending the campaign by a further channel also depends on attention costs incurred per channel. The major problem is to determine $\sum\limits_{k=1}^{n} \frac{\partial E[q_k, R|j \in I_k]}{\partial f}$ in (4.26). I show in the appendix (4.6.8) that a good approximation to this expression can be obtained by assuming that the firm competes for consumer attention with $|I| = n\phi$ firms. Then using the model of section 4.1 we get

$$\sum_{k=1}^{n} \frac{\partial E\left[q_k, R\,|j \in I_k\right]}{\partial f} \cong \frac{n\phi - R}{(n\phi)^2 f} n \tag{4.27}$$

Using the approximation (4.27) and setting $C(f) = f^\eta$ (4.26) becomes

$$y\Delta \frac{n\phi - R}{n^2 \phi A(\phi)\eta} = \kappa C(f) \tag{4.28}$$

Applying (4.28) to (4.25) and using (4.24) gives a single equation that determines ϕ for given n:

$$\frac{t\Delta}{Rn\phi}\left(1 - \frac{n\phi - R}{n\eta}\frac{A'(\phi)}{A(\phi)}\right) = A'(\phi) \tag{4.29}$$

Thus assuming an interior solution the equilibrium ϕ is approximately determined by $\psi(\phi) = 0$ where

$$\psi(\phi) \equiv \begin{cases} \frac{t\Delta}{Rn\phi}\left(1 - \frac{n\phi - R}{n\eta}\frac{A'(\phi)}{A(\phi)}\right) - A'(\phi) & R < n\phi \\ \frac{t\Delta}{(n\phi)^2} - A'(\phi) & R \geq n\phi \end{cases} \tag{4.30}$$

where the second expression follows from (2.21) and (2.22) (chapter 2.3.1).

Let $\varepsilon \equiv \frac{\phi A''(\phi)}{A'(\phi)}$ and $\nu \equiv \frac{\phi A'(\phi)}{A(\phi)}$.

Proposition 4.6. *Suppose $n > R > 1$, $A(0) < \infty$, $A'(1)/A(1) \geq 0$, $A'(1) = \infty$ as well as $\nu - \varepsilon \leq 1$.*[12] *Then the equation $\psi(\phi) = 0$ has a unique solution $\phi \in (0,1)$. If $t\Delta/R^2 - A'(R/n) > 0$ then $n\phi > R$.*

Proof: Appendix B (4.6.9)

The condition $\nu - \varepsilon \leq 1$ can be shown to hold for the advertising technology from chapter 2.1.2 (see 4.6.9 in the appendix). Henceforth I take the presuppositions of proposition 4.6 to be satisfied and assume $n\phi > R$.

The zero profit condition is $\Pi = yE[Q,R] - F - A(\phi)(1 + \kappa C(f))$. Concentrating on the case where $n\phi > R$ arises endogenously[13] and with $(1-\phi)^n \approx 0$ we get (use lemma 2.1 from chapter 2.2.2 and (4.24))

$$\frac{t\Delta}{Rn} = F + A(\phi)(1 + \kappa C(f)) \tag{4.31}$$

Using (4.28) in (4.31) gives

$$\frac{t\Delta}{Rn}\left(\frac{\eta - 1}{\eta} + \frac{R}{n\phi\eta}\right) = F + A(\phi) \tag{4.32}$$

Then equations (4.24), (4.25), (4.29) and (4.31) together determine (y, ϕ, f, n) in case of a free-entry equilibrium with $n\phi > R$. How does attention competition, i.e. the possibility to choose the salience of ones messages, alter the results from chapter 2? Looking at the equilibrium equations we see that the one parameter of attention competition that matters is η. I now compare the model of chapter one, M_1 (limited attention under informative advertising but without attention competition) to the model of this section (M_2). Let M_3 be the benchmark model of Grossman and Shapiro where limited attention is excluded by assumption. Figure 4.5 plots the equilbrium ϕ, n and f as a function of the attention cost elasticity η for all three models. In figure 4.5 the solid line corresponds to

[12] These requirements are satisfied for the CRIR advertising technology of chapter 2.

[13] The conditions for this are similar to those in chapter 2.3.2.

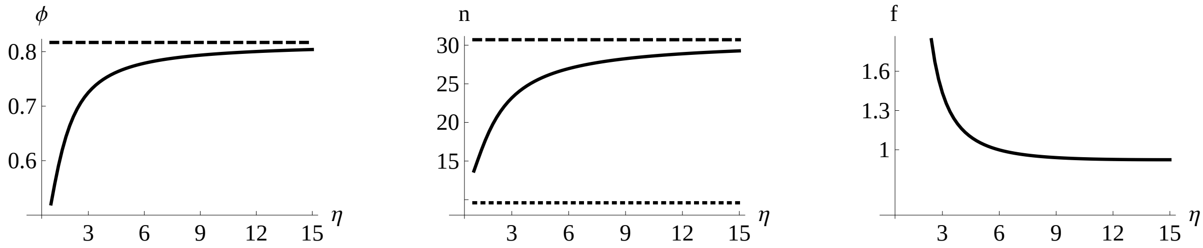

Figure 4.5: The models M_1 (dashed), M_2 (solid) and M_3 (dotted)

the solution from M_2. We see that n increases in η. We obtain a similar result in chapter 3.4.2 if the value function $V(y, y, R)$ is of the type $V(y, y, R) = \tilde{V}(y)/R$. We know from section 4.1 that the Salop model implies this type of value function. The possibility to continuously choose the reach ϕ does not change how the equilibrium n depends on η. Moreover, we note a positive relationship between ϕ and η but a negative relationship between f and η. This makes sense intuitively, because attention effort f and ϕ are somewhat substitutive: a lower f decreases the marginal costs of ϕ (see (4.25)). Comparing the solid line to the dashed line in the left picture we see that attention competition reduces equilibrium information provision. This occurs because some funds are invested into a higher salience among selected channels instead of into new channels. As attention competition imposes additional costs on firms this reduces profits compared to M_1 which explains why the level of diversity in M_2 is lower than in M_1.

4.2.3 Advertising and consumption expenditure

In the model of section 4.2 advertising expenditure consists of two components: the expenditure on reach ϕ and the expenditure on attention effort f. The Grossman and Shapiro model recognises only the expenditure on reach as variable advertising costs because unlimited attention is assumed. Since the model also predicts consumption expenditure $y\Delta$ we can construct the same ratio of advertising expenditure to consumption expenditure as was depicted in figure 1.4 of chapter 1. Let this ratio be denoted by ρ. Then

$$\rho^{LA} \equiv \frac{n^{LA}\left(A\left(\phi^{LA}\right)\left(1+\kappa C\left(f^{LA}\right)\right)\right)}{y^{LA}\Delta} \qquad \rho^{GS} \equiv \frac{n^{GS}\left(A\left(\phi^{GS}\right)\right)}{y^{GS}\Delta}$$

where the superscript LA indicates the model of section 4.2 and GS refers to the original model of Grossman and Shapiro. As Δ is exogenous to the model we can depict ρ^{LA} and ρ^{GS} as functions of Δ. From the data we know that the number of consumers in the U.S. doubled from 1950 to 2005. For fixed R we have $\rho^{LA} = \rho^{GS}$ as long as $n^{GS}\phi^{GS} \leq R$. Figure 4.6 compares the predictions of ρ by the two models to the data.[14] I normalised

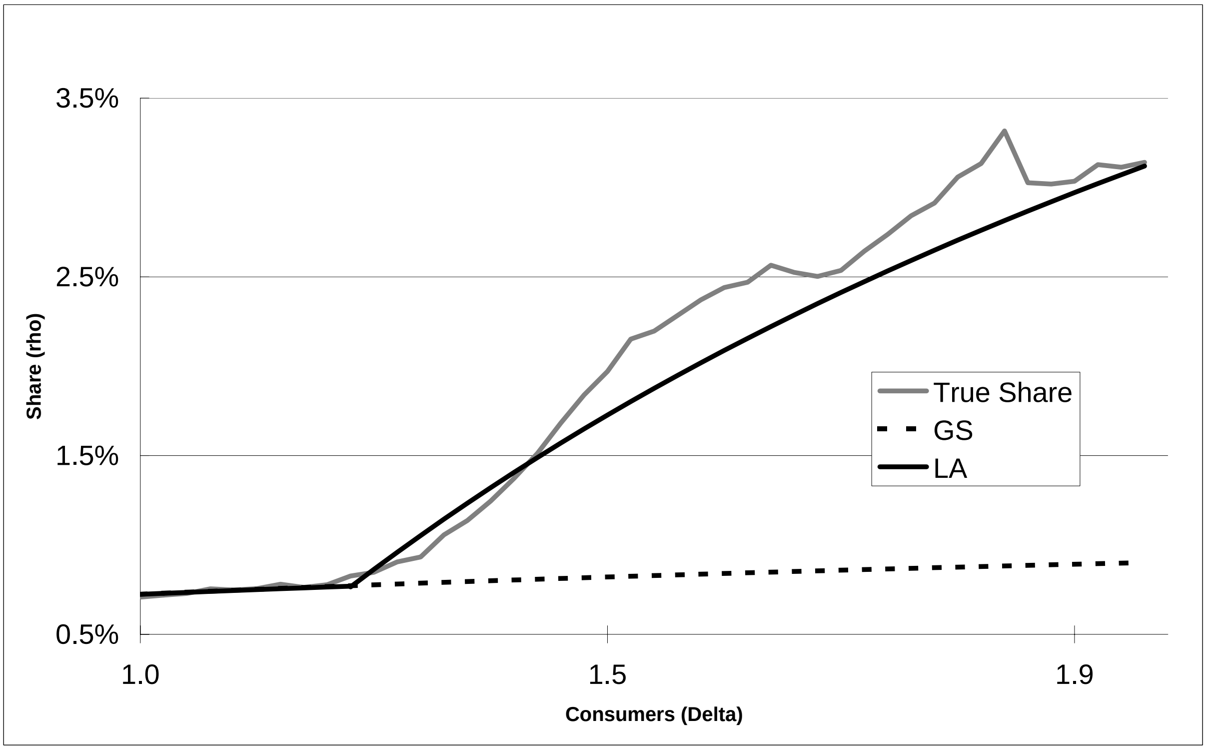

Figure 4.6: Comparison of advertising data to model predictions

initial population size to one and simulated the doubling of consumers as suggested by the data. The figure shows that the GS model as well as the LA model predict ρ to increase with Δ but at a decreasing rate. We further see that the LA model implies the share to increase at a higher rate if the attention constraint gets binding ($n\phi > R$) which leads to a much better match of the model with the data compared to the GS model. This is the case because limited attention implies more active firms and attention competition implies an additional advertising cost per firm. Moreover, this result is outstanding as the other candidate which we could expect to be important, namely advertising technology, cannot explain the remarkable increase of ρ. It can be shown that in the GS model

[14] I implemented a parameter constellation quite similar to the simulations of Grossman and Shapiro: I used the CRIR-technology with $r = 0.1$, $\theta = 0.01$, $t = 40$, $\eta = 1$, $R = 8$ and $F = 0.8$.

an increase of per channel reach r as well as a decrease of channel costs θ (both are compatible with advances in advertising technology) implies that ρ^{GS} *declines.* Hence the conventional model of GS has considerable difficulties to explain the significant increase of the advertising - consumption ratio that we observe in the data.

4.2.4 Transportation costs and informative advertising

In this section I discuss the effect of attention competition on transportation costs T in the model with informative advertising. Set $\bar{\phi} = \phi$, $\bar{y} = y$ and $\bar{f} = f$ as well as $R \geq 2$ and $n \geq 2$. Let $E[q_k, R]$ denote the probability that a consumer consumes his k-th best variety.

Lemma 4.2. *Average transportation costs conditional on being informed are*

$$T = \frac{t}{1-(1-\phi)^n} \sum_{k=1}^{n} \frac{2k-1}{4n} E[q_k, R] \tag{4.33}$$

Proof: Appendix B (4.6.10)

Note that (4.33) is valid for the model of chapter 2 as well as for the model of section 4.2. Take $\phi < 1$ and $n > 2$ as given and fixed. For simple reference let $T^L \equiv T(R < n)$ and $T^{NL} \equiv T(R \geq n)$. I now compare average transportation costs between an economy with limited attention and an economy with unlimited attention, holding all parameters other than R fixed. I prove that for given ϕ and n limited attention unambiguously implies higher average conditional transportation costs.

Proposition 4.7. *Assume $\phi \in (0, 1)$ and $n > 2$. Then $T^L > T^{NL}$.*

Proof: Appendix B (4.6.11)

Hence we get the same result regarding transportation costs in the model with informative advertising as in the model of section 4.1, where $\phi = 1$ was imposed exogenously. This occurs as limited attention implies the probability of consuming at the best locations to be lower than in the case of unlimited attention and higher at less favourable locations, which is illustrated in figure 4.7.[15] In the figure we see that $E[q_k, R]$, the probability that a consumer purchases his k-th best variety, in case of limited attention (solid line) is lower

[15]The figure has $R = 2$, $n = 10$ and $\phi = 1/2$.

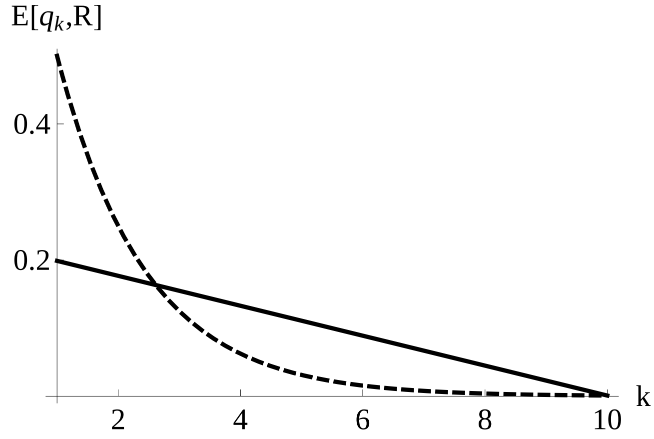

Figure 4.7: $E[q_k, R]$ for limited (solid) and unlimited attention.

for $k = 1, 2$ than in the case of unlimited attention. What is the effect of information ϕ and diversity n on transportation costs T? From Grossman and Shapiro (Grossman and Shapiro (1984), p. 74) we know that more information (a higher ϕ) or more diversity means that the average distance, $\bar{w}$, between the ideal variety and the purchased variety declines. This means that average transportation costs $T = t\bar{w}$ also decline. If consumers are informed about more varieties (higher $n\phi$) then they have a higher chance of finding a close to ideal variety. Figure 4.8 illustrates average consumer transportation costs T from (4.33) for $R = 2$, $R = 4$ and $R = \infty$ in dependence of ϕ and n. The lower set of pictures ($n = 5$ left and $n = 20$ right) shows that under limited attention more information at the firm level (higher ϕ) decreases average transportation costs too, but the effect only matters at low levels of ϕ. The upper set of pictures in figure 4.8 show that for moderate information ($\phi = 1/2$ in left picture, $\phi = 0.99$ in right picture) an increase of diversity first reduces T also in the case of limited attention and that T converges to some value that is strictly above the average distance compared to the case of unlimited attention. Moreover, if ϕ is large we observe the familiar result from section 4.1.3 that increasing diversity *increases* T.

These pictures help to understand the following result. It turns out that by using

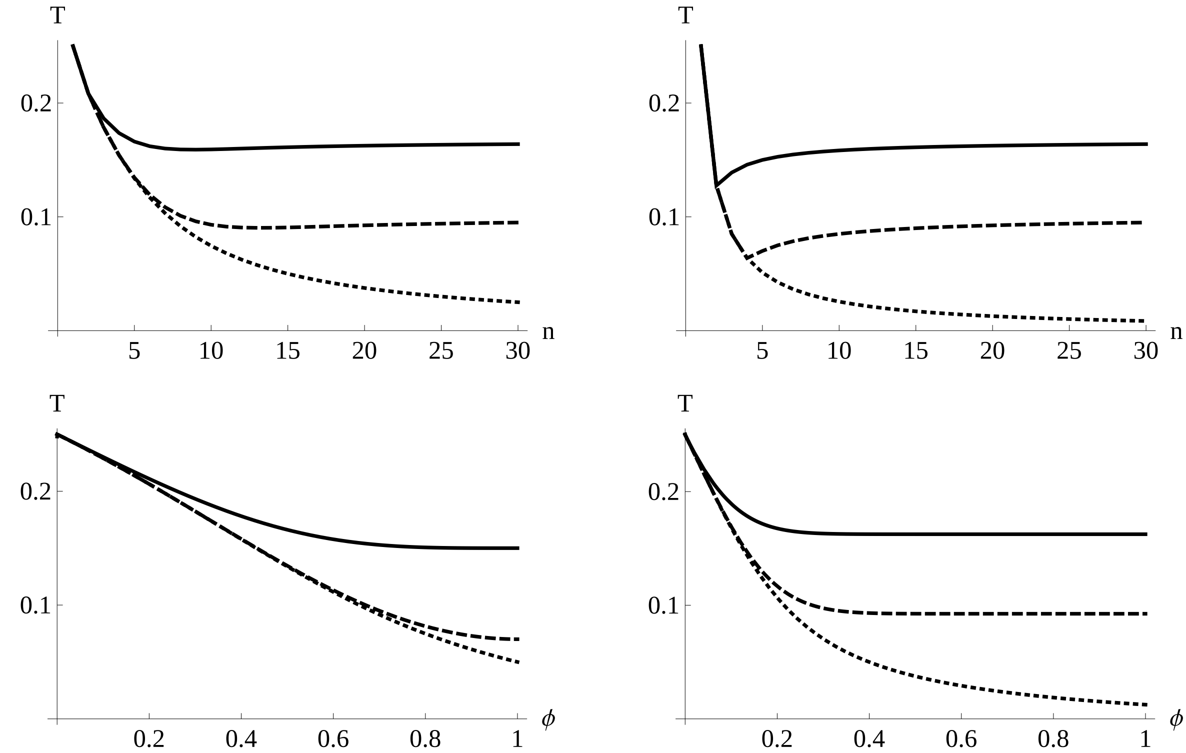

Figure 4.8: Average transportation costs: $R = 2$ (solid), $R = 4$ (dashed) and unlimited attention (dotted).

$(1-\phi)^n = 0$ in expression (4.22) we get[16]

$$E\left[q_k, R\,|j \in I_k\right] \cong \frac{R}{n\phi} \prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i} \tag{4.34}$$

The intuition of this approximation is clear. A consumer is informed of about $n\phi$ varieties. But a consumer who is aware of $n\phi > R$ varieties has the probability $p(A) = R/(n\phi)$ of choosing a particular variety and $\prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i}$ is the probability to choose the k-th best firm. For $\phi = 1$ we get $E\left[q_k, R\right]$ from (4.5). Using $E\left[q_k, R\right] = \phi E\left[q_k, R\,|j \in I_k\right]$ from (4.34) and $(1-\phi)^n = 0$ in (4.33) gives (also see lemma 4.3)

$$T \cong t \sum_{k=1}^{n} \frac{2k-1}{4n} \phi \frac{R}{n\phi} \prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i} = t\left(\frac{n+1}{2n(R+1)} - \frac{1}{4n}\right) \tag{4.35}$$

Hence we have $T = t\bar{w}^L$ where $\bar{w}^L$ is given by (4.17). This means that transportation costs

[16]These calculations are not trivial. Using Mathematica I obtained the approximation (4.34) for $(1-\phi)^n = 0$ in the cases $R = 2, 3, 4, 5$ and conjecture that (4.34) holds for any number $R > 1$.

$t\bar{w}^L$ as determined in section 4.1.3 are a reasonable approximation to the transportation costs under informative advertising whenever $(1-\phi)^n$ is close to zero (i.e. $n\phi$ is large).

From figure 4.5 we know that the effect of attention competition on the equilibrium ϕ, n are strongest if η is close to one. n and ϕ take on lower equilibrium values compared to the model of chapter 2. From figure 4.8 we know that lower ϕ ceteris paribus means higher T and lower n can mean both lower or higher T. Simulation results show that average transportation costs of the model of this chapter (M_2) are very similar to the transportation costs of chapter 4.8 under limited attention (M_1). Figure 4.9 illustrates such a simulation.[17] Moreover, transportation costs are higher compared to the case

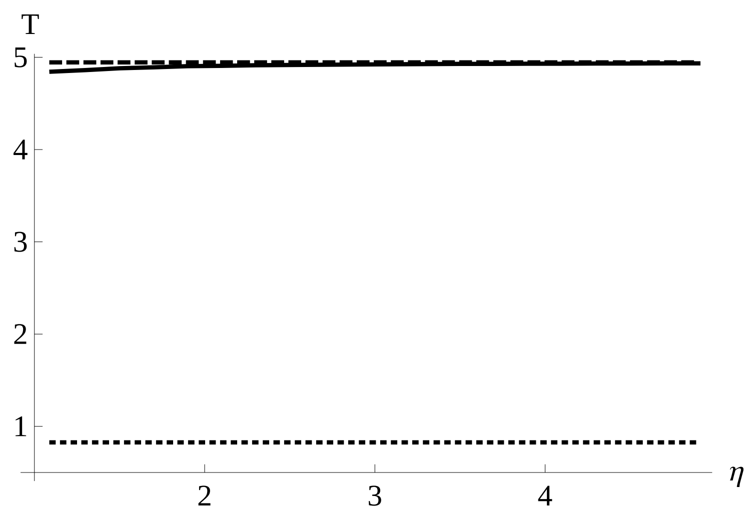

Figure 4.9: Average transportation costs for M_1 (dashed), M_2 (solid) and M_3 (dotted)

of the conventional Grossman-Shaprio model (M_3). This means that the main welfare conclusion from section 4.1.3, limited attention increases transportation costs (consumer-firm mismatch) compared to the conventional model, is not only valid in the Internet economy but also when firms can continuously choose the reach of their information campaign.

[17] This particular figure has $R = 2$, $t = v = 30$, $\theta = 0.01$, $F = 0.3$ and uses the CRIR-technology with $r = 1/3$. All simulations that I have conducted reveal a similar picture.

4.3 Negative externality of limited attention among consumers?

Up to now I have assumed $R_i = R$ for all consumers. Suppose now there are two types of consumers with respect to the size of their attention set. A fraction $\lambda \in (0,1)$ of all consumers has R_L and a fraction of $1-\lambda$ has R_H with $R_L < R_H$. Then $\bar{R} = \lambda R_L + (1-\lambda)R_H$ is the average attention threshold in the economy. We can think of the L-types as very inattentive shoppers whereas the H-type represents a more careful type of consumer who considers more varieties in making his decision. In the model of section 4.1 (attention competition in the Salop model) the profit function of the representative firm becomes

$$\begin{aligned}\Pi &= y\left(\lambda p(f,\bar{f},n,R_L)\sum_{k=1}^{n} p\left(k, R_L\left|A\right.\right)N_k + (1-\lambda)p(f,\bar{f},n,R_H)\sum_{k=1}^{n} p\left(k, R_H\left|A\right.\right)N_k\right)\\ &\quad -F - C(f)\end{aligned} \tag{4.36}$$

where $p\left(k, R_\varsigma\left|A\right.\right)$ denotes the conditional probability of offering the highest net utility among all perceived alternatives to a ς-type member of group k. We can use proposition 4.2 to conclude that if $\sqrt{t/F} > \bar{R}$ then an equilibrium with $n > \bar{R}$ occurs. Assuming an interior equilibrium with $R_L < \bar{R} < R_H < n$, using the ACF and proceeding as in section 4.1.2 the following three equilibrium conditions can be derived from (4.36):

$$y = \frac{t}{\bar{R}} \tag{4.37}$$

$$\frac{t(n-\bar{R})}{\bar{R}n^2 f} = C'(f) \tag{4.38}$$

$$\frac{t}{\bar{R}n} = F + C(f) \tag{4.39}$$

We see that only average attention $\bar{R}$ matters for the equilibrium. Comparing these equations with (4.13) we note that the comparative statics of (y, f, n) are similar to those of section 4.1.2.1. With $C(f) = \theta f^\eta$ and by using (4.38) in (4.39) we get a quadratic equation in n which has the positive solution

$$n = \frac{(\eta - 1)t + \sqrt{t^2(\eta-1)^2 + 4tF\bar{R}^2\eta}}{2F\bar{R}\eta} \tag{4.40}$$

Equilibrium utility of the H-type is

$$U_H = v - \frac{t}{\bar{R}} - tT_H \tag{4.41}$$

where T_H are average transportation costs that an H-type incurs. Now suppose that[18] $dR_L < 0$ such that $d\bar{R} < 0$. What is the effect of such a change on U_H? First, we see that the price increases ($y'(\bar{R}) < 0$) which decreases U_H. If $R_H < n$ then, because $T_H'(n) > 0$ and $n'(\bar{R}) < 0$, T_H increases as R_L decreases. This means that U_H decreases if R_L decreases. Less attention of the L-type ceteris paribus hurts the H type (who is also attention constrained as $n > R_H$ because of higher prices and higher transportation costs. Suppose now that in equilibrium we have $R_L < \bar{R} < n < R_H$. Then we have $T_H(n) = t/(4n)$ and $T_H'(n) < 0$ (follows from section 4.1.3). The H type perceives the entire market. Hence if diversity increases marginally (so that the H type ex post still perceives the entire market) then the H-type will benefit from this because his transportation costs unambiguously decrease in n. However, equilibrium prices are higher if $dR_L < 0$. The question is whether the increase in prices can be at least compensated by the decrease in transportation costs. Suppose that λ is close to one. Hence there are only few H-types and their impact on equilibrium is negligible so that equations (4.37) - (4.39) remain valid. Using (4.41) and (4.40) we have

$$(\eta - 1)^2 \leq (\eta - 9)\sqrt{(\eta-1)^2 + \frac{4F\bar{R}^2\eta}{t}}$$

But as $\bar{R} < n$ requires that $F < t/\bar{R}^2$ we have

$$(\eta - 9)\sqrt{(\eta-1)^2 + \frac{4F\bar{R}^2\eta}{t}} < (\eta - 9)\sqrt{(\eta-1)^2 + 4\eta} = (\eta - 9)(\eta + 1)$$

[18]Equivalently, we could also set $d\lambda > 0$.

As $(\eta-1)^2 > (\eta-9)(\eta+1)$ we get a contradiction. Hence also in the case where $R_H > n$ we have that the utility of the H-type decreases if R_L decreases, at least in the case with few H-types. The L-type exhibits an indirect negative externality on the H-type because prices are determined by average attention and are higher as if only the H-type was present. Moreover, if there are only few H-types with $R_H > n$ there is not enough diversity to compensate the H-type for the higher prices the have to pay.

4.4 Adverse effects of attentional advantages?

As we have seen under limited attention there is an interaction between the competition for attention and the economic competition within an attention set. Up to now firms have been symmetrically endowed both with production and attention technology.

An interesting question is to explore what happens if we allow for asymmetry in both types of technologies. To illustrate possible implications of this kind of asymmetry I use the ACF and the linear demand function from section 3.6.2 in case of $R=2$. I introduce asymmetry as simply as possible by assuming that firm $j=1$ is different from all other active firms. I consider firm $j=2$ to be a representant of this other class.I assume constant unit costs of production c_j and $C_j(f_j)=\theta_j f^\eta$. Firm $j=1$ differs from the other firms in unit costs c_1 and attention costs θ_1. We can write the reduced form of the profit functions of this problem as

$$\Pi^1 = \left(1-\prod_{i=1}^{2}\frac{f_2(n-i)}{f+f_2(n-i)}\right)(y_1-c_1)\left(\frac{1-y_1-\gamma(1-y_2)}{1-\gamma^2}\right)-\theta_1 f_1^\eta$$

$$\Pi^2 = (y_2-c_2)\left(P_a\frac{1-y_2-\gamma(1-y_1)}{1-\gamma^2}+P_b\frac{1-y_2-\gamma(1-\tilde{y})}{1-\gamma^2}\right)-\theta_2 f_2^\eta$$

where

$$\begin{aligned} P_a &= \tfrac{f_1 f_2}{f_1+f_2+(n-2)\tilde{f}}\left(\tfrac{1}{f_1+(n-2)\tilde{f}}+\tfrac{1}{f_2+(n-2)\tilde{f}}\right)\\ P_b &= \tfrac{f_2\tilde{f}}{f_1+f_2+(n-2)\tilde{f}}\left(\tfrac{n-2}{f_1+(n-2)\tilde{f}}+\tfrac{n-2}{f_1+f_2+(n-3)\tilde{f}}\right)\end{aligned}$$

are the probabilities of firm $j=2$ to be in an attention set with either $j=1$ or with any other firm (all other firms set $(\tilde{y},\tilde{f})$). The equilibrium values of (y_1,y_2,f_1,f_2) are then given by the two sets of FOC's evaluated at $\tilde{y}=y_2$ and $\tilde{f}=f_2$. The first figure illustrates the equilibrium as a function of n with $\gamma = 1/2$, $F = 0.01$, $\theta_1 = \theta_2 = 1/4$, $\eta = 1$, $c_1 = 0.2$ and $c_2 = 0.1$. In figure 4.10 we see that the cost advantages ($c_2 < c_1$) of the firms of type

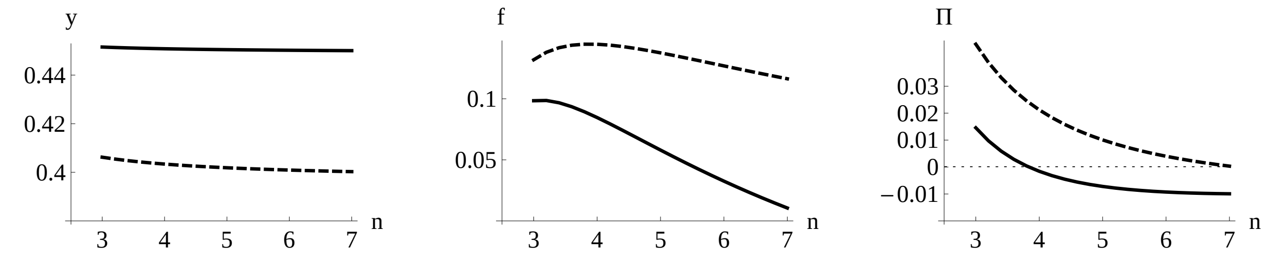

Figure 4.10: Firm $j = 1$ (solid) and firm $j = 2$: $c_2 < c_1$ and $\theta_1 = \theta_2$

$j = 2$ imply that these firms can sustain a lower price in equilibrium. This implies a higher marginal return on attention and thus $f_2 > f_1$ and also $\Pi_2 > \Pi_1$. The cost advantage of the other firms implies that firm $j = 1$ has a lower chance of getting attention as it cannot afford the required high attention costs because it earns less revenue from every set it is in. Thus under free-entry such a firm would not survive.

The story may change entirely if $c_2 < c_1$ but $\theta_1 < \theta_2$, i.e. firm $j = 1$ has lower cost of attention effort f. The next figure is obtained by setting $\theta_1 = 1/10$. The other parameters have the same value as before. Firm $j = 1$ still is less efficient in production which implies

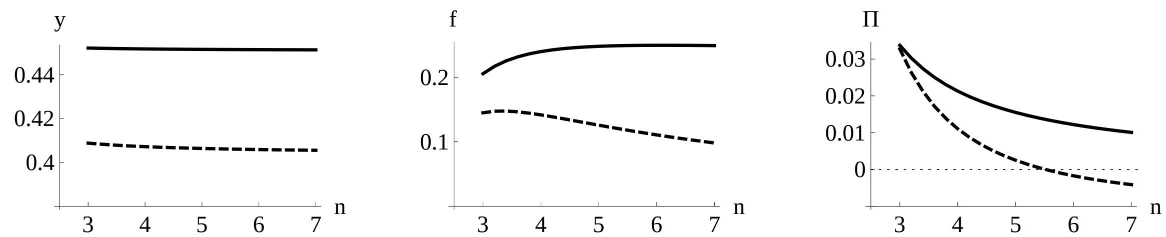

Figure 4.11: Firm $j = 1$ (solid) and firm $j = 2$: $c_2 < c_1$ and $\theta_1 < \theta_2$

that it must set higher prices to cover production costs. At the same time the firm has lower costs of maintaining a certain level of attention, e.g. because of better knowledge of marketing instruments, which means that such a firm exhibits more attention effort. Despite being more expensive such a firm may achieve to be in many more attention sets which compensates for the loss of demand due to higher prices. Hence under free entry such a firm survives and to some extent crowds-out firms with more efficient production methods. Also note that in this example the higher attention effort of the firm does not imply its offers to be superior - in fact they are more expensive than other comparable offers. Nelson argues that with experience goods "advertised brands are better buys" and moreover that "heavily advertised brands are likely to provide a lower P^*" (price per unit of utility) because these firms can afford higher advertising expenditure due to lower production costs (Nelson (1974), p. 732). This negative association between advertising and pricing is reflected[19] in figure 4.10. However, as figure 4.11 shows there can be positive association between advertising expenditure at the firm level and price because under limited attention a firm may effectively suppress the perception of other potentially cheaper firms to some degree which makes a higher price sustainable. Hence in contrast to Nelson's conjecture more advertising by a firm need not signal a superior product. Figure 4.12 shows that such an advantage in advertising technology may act as an entry barrier that crowds out potentially superior producers. The figure depicts

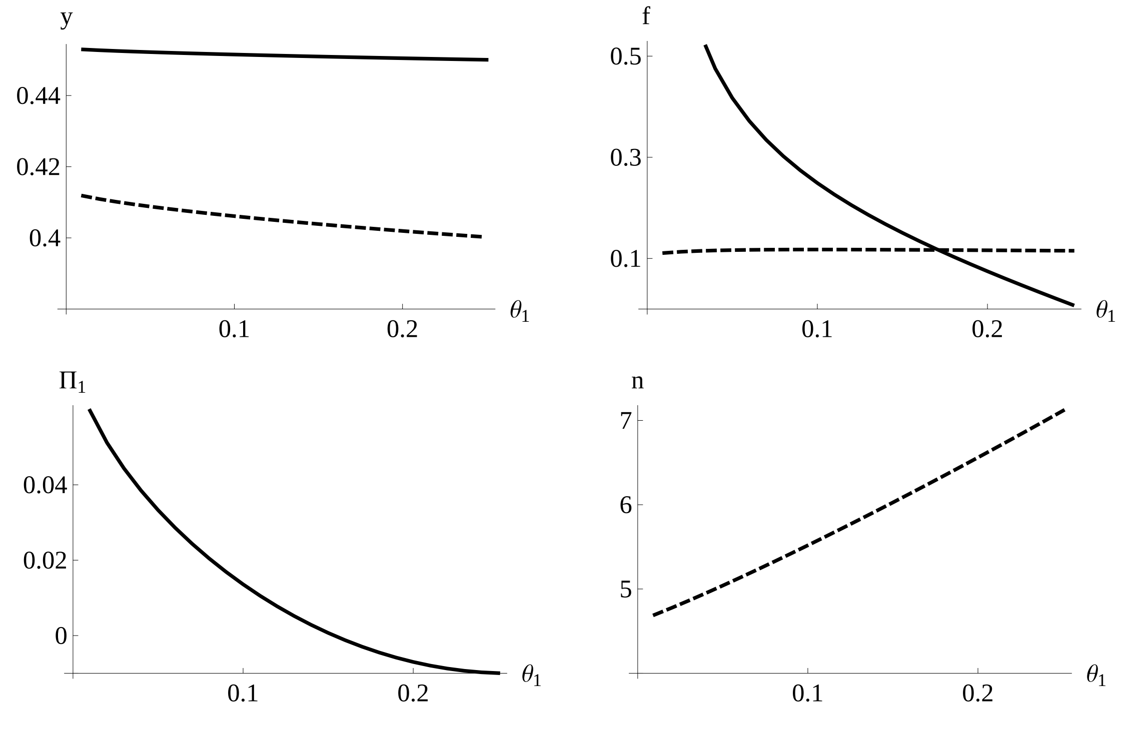

Figure 4.12: Firm $j = 1$ (solid) and firm $j = 2$

the equilibrium variables under endogenous n as a function of θ_1. All other parameters take on the same values as before. If θ_1 is sufficiently low then firm $j = 1$ survives. An advantage in marketing implies that the firm can sustain a higher level of f. This makes the firm present in more attention sets and compensates for the necessity to set higher

[19]As in this example all commodities symmetrically enter the utility function changes of y_j also reflect changes of utility adjusted prices.

prices because of higher production costs. Moreover, the other firms are urged to increase their attention efforts which increases their costs and reduces profits such that some firms must leave the market. Finally, under the assumption that all potential entrants to the market are of type $j = 2$ firm $j = 1$ manages to earn a positive rent also under free entry.

By taking two snapshots of a subset of the web Cho and Roy (2004) investigate how the popularity of the pages measured by the number of incoming links in that subset evolves over a period of seven months. They find that the top 20 percent of the pages with most incoming links collect 70 percent of the new links throughout the 7 months while the bottom 60 percent of the pages obtained practically no new links (p. 22). The basic explanation of this observation presented by the authors is that the use of search-engines dominates the browsing behaviour of web clients. Moreover, they estimate that a new and hence initially unpopular page of high quality (a page which most users like if they find it) takes 66-times longer to become popular if clients use a search-engine instead of just following links. The reason for this is that search engines based on a PageRank algorithm direct only little traffic to unpopular pages (p. 27). In the light of the previous result on asymmetric capabilities to attract attention this empirical finding strongly suggest not to underestimate the implications of attention competition: limited attention and the strong usage of search engines imply that established firms with the possibility to achieve higher attention by means of a better linkage may effectively avoid some economic competition. Firms with cheaper offers that are less established are less perceived and, different from the standard economic prediction, cannot translate there cost advantages into revenues. This in the end might hurt the consumers.

4.5 Appendix A

4.5.1 More on transportation costs

Section 4.1.3 formally establishes a positive relationship between transportation costs (transportation distance) and the degree of diversity n under limited attention and provides an intuition for the result. In this section I dig deeper and illustrate that this result is caused because under limited attention the *entire* distribution function of transportation distance over the population tends to shift downwards under limited attention.

Consider the n-sequence of intervals

$$O(n) = \left\{ \left[\frac{k-1}{2n}, \frac{k}{2n} \right) \right\}_{k=1}^{n-1} \cup \left[\frac{n-1}{2n}, \frac{1}{2} \right]$$

Let $O_k(n)$ denote the kth element of $O(n)$ and let $w \in [0, 1/2)$ denote travel distance. Then because $O(n)$ forms a partition of $[0, 1/2]$ we have $\exists! \tilde{k} \in \{1, 2, ..., n\}$ such that $w \in O_{\tilde{k}}(n)$. Hence $2nw < \tilde{k} \leq 2nw + 1$. Because consumers are uniformly distributed over the circle the density $g(w)$ of travel distance $w \in [0, 1/2)$ is given by

$$g(w) = E[q_k, R]\gamma(w) \quad w \in O_k(n)$$

where $E[q_k, R]$ is the probability to consume the k-th favourite variety and $\gamma(w)$ is the density of w for $w \in O_k(n)$. Because consumers are uniformly distributed over the circle we have $\gamma(w) = \frac{1}{\frac{k}{2n} - \frac{k-1}{2n}} = 2n$ for $k = 1, ..., n$. Hence the fraction of people who travel a distance of less than $w \in [0, 1/2)$ is

$$\begin{aligned} G_n(w) &= \int_0^{1/(2n)} E[q_1, R]2n\, ds + \int_{1/(2n)}^{1/n} E[q_2, R]2n\, ds + ... \\ &+ \int_{(k-1)/(2n)}^{k/(2n)} E[q_k, R]2n\, ds + ... + \int_{([2wn]-1)/(2n)}^{[2wn]/(2n)} E[q_{[2wn]}, R]2n\, ds + \int_{[2wn]/(2n)}^{w} E[q_{[2wn]+1}, R]2n\, ds \\ &= \sum_{k=1}^{[2wn]} E[q_k, R] + \int_{\frac{[2wn]}{2n}}^{w} E[q_{[2wn]+1}, R]2n\, ds \end{aligned} \tag{4.42}$$

where $[x] \leq x$ denotes the next lower integer to x. To see how $G_n(w)$ depends on n let $n' = 2n$ and $n \geq 2$. Suppose first that $n' \leq R$. Then

$$G_n(w) = \begin{cases} 2nw & w \in [0, 1/(2n)) \\ 1 & w \in [1/(2n), 1/2] \end{cases} \qquad G_{n'}(w) = \begin{cases} 4nw & w \in [0, 1/(4n)) \\ 1 & w \in [1/(4n), 1/2] \end{cases}$$

Hence $G_n(w)$ first-order stochastically dominates $G_{n'}(w)$: This means that in the absence of limited attention the fraction of people travelling at most a certain distance w can never increase (but decreases over a certain range) if diversity is increased. This finding is a

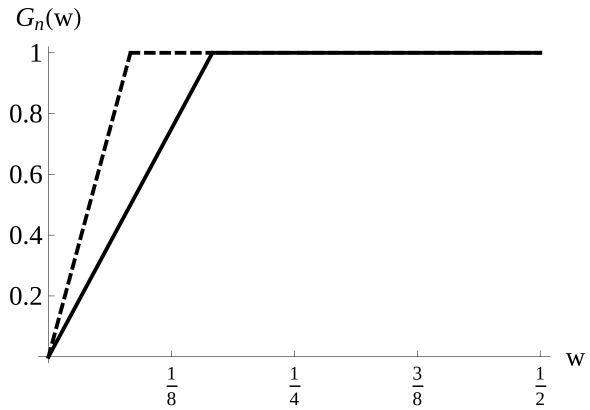

Figure 4.13: $G_3(w)$ (solid) and $G_6(w)$ with $R > n'$

reinforcement of the fact that $T'(n) < 0$ if $n < R$ as stochastic dominance always implies mean dominance. As figure 4.14 illustrates the opposite is true if $R < n < n' = 2n$. As

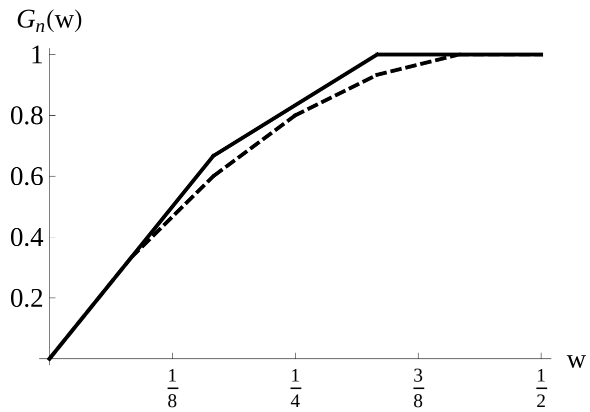

Figure 4.14: $G_3(w)$ (solid) and $G_6(w)$ with $R = 2$

the figure suggests doubling diversity under limited attention implies that the fraction of consumers that travel at most a certain distance w is never larger under limited attention but sometimes smaller. Transportation costs are given by $T(w) = tw \in [0, t/2]$. For example, from figure 4.14 we see that for $n = 3$ approximately 80% of the consumers incur transportation costs of $t/4$ or less. For $n = 6$ only about 75% of the population has costs of $t/4$ or less, the other 25% have higher costs. Hence the figure suggests that average transportation costs $T = t\bar{w}$ increase under limited attention if diversity increases *because* this shifts at least a part of the distribution function $G(n)$ downwards.

4.6 Appendix B

4.6.1 Proof of lemma 4.1

Note that $v - t/2 - y \geq 0$ for $\bar{y} = y \leq v - t/2$ implies that all consumers purchase somewhere as $1/2$ is the maximal distance a consumer must travel.

a)

$$p\left(1\,|A\right) = \prod_{i=1}^{R-1} \frac{n-i}{n-i} = 1$$

$$p\left(n\,|A\right) = \prod_{i=1}^{R-1} \frac{1-i}{n-i} = 0$$

b) For $n > R$ we have $N_k = 1/n$ for $k = 1, ..., n$ as $y = \bar{y}$. Use use (4.4) and lemma 4.3 (see below) to get

$$V(y, y, R) = y \sum_{k=1}^{n} p\left(k\,|A\right) N_k = y \frac{1}{n} \sum_{k=1}^{n} \prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i} = \frac{y}{n} \frac{n}{R} = \frac{y}{R}$$

For $R \geq n$ we have $V(y, y, n) = yN_1 = \frac{y}{n}$

c) For $f = \bar{f} > 0$ and $n > R$ we have $p(A) = R/n$ (see lemma 3.3). Hence

$$E\left[Q, z\right] = \begin{cases} \frac{R}{n} \sum\limits_{k=1}^{n} p\left(k\,|A\right) N_k & R < n \\ N_1 & R \geq n \end{cases}$$

Hence if $n > R$ we have (use (4.4), lemma 4.3 below)

$$\begin{aligned} E\left[Q, R\right] &= \tfrac{R}{n} \sum_{k=1}^{n} p\left(k\,|A\right) N_k = \tfrac{R}{n} \sum_{k=1}^{n} \prod_{i=1}^{R-1} \tfrac{1+n-k-i}{n-i} \tfrac{1}{n} \\ &= \tfrac{R}{n^2} \sum_{k=1}^{n} \prod_{i=1}^{R-1} \tfrac{1+n-k-i}{n-i} = \tfrac{R}{n^2} \tfrac{n}{R} = \tfrac{1}{n} \end{aligned}$$

If $R \geq n$ then $E\left[Q, n\right] = N_1 = \frac{1}{n}$.

$\blacksquare$

Lemma 4.3. *For $n > R$ we have*

$$\sum_{k=1}^{n} p\left(k\,|A\right) = \sum_{k=1}^{n}\prod_{i=1}^{R-1}\left(\frac{1+n-k-i}{n-i}\right) = \frac{n}{R} \tag{4.43}$$

Proof: The proof is by induction over n. Take an arbitrary $R > 1$. For the base step set $n = R + 1$. Then

$$\prod_{i=1}^{R-1}\frac{2+R-k-i}{R+1-i} = 0$$

for $k > 2$ since there is $i \in \{1, ..., R-1\}$ so that $i = 2 + R - k$. Hence

$$\sum_{k=1}^{R+1}\prod_{i=1}^{R-1}\tfrac{2+R-k-i}{R+1-i} = \sum_{k=1}^{2}\prod_{i=1}^{R-1}\tfrac{2+R-k-i}{R+1-i}$$

$$= 1 + \tfrac{1}{R} = \tfrac{1+R}{R}$$

because

$$\prod_{i=1}^{R-1}\frac{R-i}{R+1-i} = \frac{R-1}{R}\frac{R-2}{R-1}\cdot ....\cdot\frac{2}{3}\cdot\frac{1}{2} = \frac{1}{R}$$

This proves the base step. Now assume (4.43) is true. Hence

$$\sum_{k=1}^{n}\prod_{i=1}^{R-1}(1+n-k-i) = \frac{n}{R}\prod_{i=1}^{R-1}(n-i)$$

Then

$$\begin{aligned}
&\sum_{k=1}^{n+1}\prod_{i=1}^{R-1}\tfrac{2+n-k-i}{1+n-i} = \prod_{i=1}^{R-1}\tfrac{1}{1+n-i}\left(\sum_{k=1}^{n+1}\prod_{i=1}^{R-1}(2+n-k-i)\right)\\
&= \prod_{i=1}^{R-1}\tfrac{1}{1+n-i}\left(\sum_{k=0}^{n}\prod_{i=1}^{R-1}(1+n-k-i)\right)\\
&= \prod_{i=1}^{R-1}\tfrac{1}{1+n-i}\left(\sum_{k=1}^{n}\prod_{i=1}^{R-1}(1+n-k-i) + \prod_{i=1}^{R-1}(1+n-i)\right)\\
&= \prod_{i=1}^{R-1}\tfrac{1}{1+n-i}\left(\tfrac{n}{R}\prod_{i=1}^{R-1}(n-i) + \prod_{i=1}^{R-1}(1+n-i)\right) = \left(\tfrac{n}{R}\prod_{i=1}^{R-1}\tfrac{(n-i)}{1+n-i} + 1\right)\\
&= \tfrac{n}{R}\tfrac{1+n-R}{n} + 1 = \tfrac{1+n}{R}
\end{aligned} \tag{4.44}$$

■

4.6.2 Proof of proposition 4.1

Note that the formulas for N_k from chapter 2 are valid as long as all consumers consume somewhere. This is the case if his net utility from consumption is non-negative, i.e. if $u \geq 0$. As $R, n \geq 2$ we see from (4.8) and (4.11) that $v \geq t$ implies that $u_i \geq 0$ because maximal transportation distance is 1/2. Let $n > R$. Assuming an interior solution we can evaluate $\Pi_y = 0$ and $\Pi_f = 0$ at $\bar{y} = y > 0$ (derived from (4.7)) and $\bar{f} = f > 0$ which gives (use lemmata 4.3 and 4.1 and (4.6))

$$\begin{array}{ll} \Pi_y = 0 \quad \Rightarrow & E[Q,R] + y\frac{R}{n}\left(\frac{-1}{t}\right) = 0 \\ \Leftrightarrow \frac{1}{R} = \frac{y}{t} \quad \Leftrightarrow & y = t/R \end{array}$$

and

$$\Pi_f = 0 \quad \Rightarrow \quad p_1(f,f,n,R)V(y,y,R) = C'(f) \quad \Leftrightarrow \quad p_1(f,f,n,R)\frac{t}{R^2} = C'(f)$$

where I used $y = t/R$ in the last part. Second-order conditions are satisfied at the equilibrium because

$$\frac{\partial^2 \Pi}{\partial y^2} = \frac{2R}{nt}\left(p\left(n\,|A\right) - p\left(1\,|A\right)\right) = -\frac{2R}{nt}$$

$$\frac{\partial^2 \Pi}{\partial f^2} = p_{11}(f,f,n,R)\frac{t}{R^2} - C''(f) < 0$$

$$\frac{\partial^2 \Pi}{\partial y \partial f} = p_1(f,f,n,R)\left(\frac{1}{R} - \frac{y}{t}\right) = 0$$

Obviously, equilibrium prices are uniquely determined in the symmetric equilibrium and (4.9) has a solution with $f \in (0, \infty)$ by the properties of the ACF (see chapter 3.3.1). If $R \geq n$ then we get the conventional Salop equilibrium. (4.10) and (4.12) are obvious.

■

4.6.3 Proof of proposition 4.2

Because the symmetric opponent form in (4.7) is a special case of (3.9) (or (3.10)) we can apply proposition 3.4 (chapter 3). As we use the ACF we need only verify whether assumptions 3.5 and 3.6 are satisfied. Because we can explicitely calculate y (see (4.8) and (4.11)) assumption 3.5 is trivially satisified. Because $V(y, y, z) = y/z$ assumption 3.6 is also satisfied. As $\hat{y} = t/2$ and $V(\hat{y}, \hat{y}, 2) = t/4$ and $\Delta = 1$ condition (3.27) becomes $\frac{t}{4F} \geq 1$ which is a presupposition of proposition 4.2. Consequently, we may conclude that a single symmetric SPE (y, f, n) exists with $n \geq 2$ by proposition 3.4. According to (3.32) (see chapter 3.4.2) and (4.14) a conventional equilibrium requires that $\tilde{n} = \sqrt{\frac{t}{F}} \leq R$. Thus if $\sqrt{\frac{t}{F}} \leq R$ then the equilibrium (y, f, n) is determined by $\left(\sqrt{tF}, 0, \sqrt{t/F}\right)$. If we have $\sqrt{\frac{t}{F}} > R$ then an attention equilibrium occurs and (y, f, n) is determined by (4.13). The second equation of (4.13) follows from using the approximation (3.21) in(4.9).

$\blacksquare$

4.6.4 Proof of proposition 4.3

Let $n > R$. Note that

$$\sum_{k=1}^{n} E[q_k, R]\bar{w}_k = \sum_{k=1}^{n} E[q_k, R]\frac{2k-1}{4n} = \frac{1}{4n}\left(2\sum_{k=1}^{n} E[q_k, R]k - \sum_{k=1}^{n} E[q_k, R]\right)$$

For $\bar{f} = f$ (4.5) becomes

$$E[q_k, R] = \frac{R}{n}\prod_{i=1}^{R-1}\left(\frac{1+n-k-i}{n-i}\right)$$

But (use lemma 4.3 (4.6.1))

$$\sum_{k=1}^{n} E[q_k, R] = \frac{R}{n}\sum_{k=1}^{n}\prod_{i=1}^{R-1}\left(\frac{1+n-k-i}{n-i}\right) = \frac{R}{n}\frac{n}{R} = 1$$

Step 1:

I first prove that

$$\sum_{k=1}^{n} E[q_k, R]k = \frac{(n+1)}{(R+1)} \tag{4.45}$$

by induction over n. For base step set $n = 1 + R$. We need to show that

$$\sum_{k=1}^{R+1} E[q_k, R]k = \frac{R+2}{R+1}$$

But

$$\sum_{k=1}^{R+1} E[q_k, R]k = \frac{R}{R+1} \sum_{k=1}^{R+1} \prod_{i=1}^{R-1} \frac{(2+R-k-i)k}{1+R-i}$$

It is not hard to see that $\prod_{i=1}^{R-1} \frac{(2+R-k-i)k}{1+R-i} = 0$ for $k > 2$. Hence

$$\sum_{k=1}^{R+1} E[q_k, R]k = \frac{R}{R+1} \sum_{k=1}^{2} \prod_{i=1}^{R-1} \frac{(2+R-k-i)k}{1+R-i} = \frac{R}{R+1}(1+\frac{2}{R}) = \frac{2+R}{R+1}$$

which completes the base step. Suppose (4.45) is true. Then

$$\sum_{k=1}^{n} \prod_{i=1}^{R-1} (1+n-k-i)k = \frac{n(1+n)}{R(R+1)} \prod_{i=1}^{R-1} (n-i) \tag{4.46}$$

Now

$$\sum_{k=1}^{n+1} \prod_{i=1}^{R-1} \frac{(2+n-k-i)k}{1+n-i} = \prod_{i=1}^{R-1} \frac{1}{1+n-i} \left(\sum_{k=1}^{n+1} \prod_{i=1}^{R-1} (2+n-k-i)k \right)$$

$$= \prod_{i=1}^{R-1} \frac{1}{1+n-i} \left(\sum_{k=0}^{n} \prod_{i=1}^{R-1} (1+n-k-i)(k+1) \right)$$

$$= \prod_{i=1}^{R-1} \frac{1}{1+n-i} \left(\sum_{k=1}^{n} \prod_{i=1}^{R-1} (1+n-k-i)k + \sum_{k=0}^{n} \prod_{i=1}^{R-1} (1+n-k-i) \right)$$

$$= \frac{n(1+n)}{R(R+1)} \prod_{i=1}^{R-1} \frac{(n-i)}{1+n-i} + \sum_{k=0}^{n} \prod_{i=1}^{R-1} \frac{(1+n-k-i)}{1+n-i}$$

$$= \frac{n(1+n)}{R(R+1)} \frac{1+n-R}{n} + \frac{1+n}{R} = \frac{(1+n)(2+n)}{R(R+1)}$$

where the last line uses (4.44).

Step 2:

Hence

$$\sum_{k=1}^{n} E[q_k, R]\frac{2k-1}{4n} = \frac{1}{4n}\left(2\sum_{k=1}^{n} E[q_k, R]k - \sum_{k=1}^{n} E[q_k, R]\right)$$

$$= \frac{1}{4n}\left(\frac{2(n+1)}{(R+1)} - 1\right) = \frac{(n+1)}{2n(R+1)} - \frac{1}{4n}$$

The other claims are obvious.

■

4.6.5 Proof of proposition 4.4

Suppose $T(n) = t/(4n)$ for all $n \geq 1$. In this case the solution of the planer problem is given by $\tilde{n} = \frac{1}{2}\sqrt{\frac{t}{F}}$ Hence if $\tilde{n} \leq R$ then $n^P = \tilde{n}$. If $\tilde{n} > R$ then by (4.19) $\tilde{n}$ cannot be a solution to the planer problem. Because $T'(n) > 0$ for $n > R$ the planer would never choose $n^P > R$. But $n^P < R$ could not be optimal because then $W'(n^P) > 0$. Hence we must have $n^P = R$.

■

4.6.6 Proof of proposition 4.5

For the first claim see the main text. Now assume $\eta = 1$. In this case we have $n = \sqrt{t/F}$ which corresponds to the conventional Salop solution. Then if $n^p = \frac{1}{2}\sqrt{t/F}$ we have $\partial n^P/\partial R = 0$ and hence $\rho'(R) = 0$. Thus excessive diversity in this case is the same with or without limited attention. If however $n^P = R$ we have $\rho'(R) = -1 < 0$. If $\eta > 1$ we always have $\rho'(R) < 0$ because $n'(R) < 0$ and $\partial n^P/\partial R \geq 0$. See the main text for the last claim.

■

4.6.7 Deriavtion of (4.21)

Suppose a member of group k is informed about the representative firm and $z \geq R$ opponents. Hence $|I_k| = 1 + z > R$. Suppose that I_k contains exactly $s \leq min\{k-1, z\}$ superior firms (and hence $z - s$ inferior firms). The probability that this event occurs is exactly given by G_1 in (4.21) (see also 2.5.3 in the appendix of chapter 2). Let $P_{k,s,z}$ denote the probability, conditional on $j \in I_k$, to make a sale to a member of group k if s superior firms are in I_k and the consumer received information of z opponents. Using (4.2) - (4.4) we see that for given k, z and s we have

$$\begin{aligned} P_{k,s,z} &= \underbrace{p\left(f, \bar{f}, 1+z, R\right)}_{p(A)} \underbrace{\prod_{i=1}^{R-1} \left(1 - \frac{s\bar{f}}{(1+z-i)\bar{f}}\right)}_{p(k|A)} \\ &= p\left(f, \bar{f}, 1+z, R\right) \prod_{i=1}^{R-1} \tfrac{1+z-i-s}{(1+z-i)} \end{aligned} \tag{4.47}$$

where $p(A)$ is the representative firm's probability to be in the attention set and $p(k\,|A)$ is the probability to be the superior perceived firm. Note that if $s \leq 1 + z - R$ we have $p(k\,|A) > 0$. A consumer who perceives the representative firm ignores $1 + z - R$ other firms. If $s \leq 1 + z - R$ then there is a positive probability that the consumer overlooks precisely the s superior firms. If however $s > 1 + z - R$ then this is not possible and $p(k\,|A) = 0$. By taking the sum over $s = 0, ..., k-1$ we calculate the conditional probability that the representative makes a sale to a member of group k given that this consumer has received information from $z \geq R$ opponents:

$$\begin{aligned} P_k(S(z)\,|z \geq R) &= \sum_{s=0}^{k-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \left(\begin{array}{c} k-1 \\ s \end{array}\right) \left(\begin{array}{c} n-k \\ z-s \end{array}\right) P_{k,s,z} \\ &= \sum_{s=0}^{k-1} \bar{\phi}^z (1-\bar{\phi})^{n-1-z} \left(\begin{array}{c} k-1 \\ s \end{array}\right) \left(\begin{array}{c} n-k \\ z-s \end{array}\right) p\left(f, \bar{f}, 1+z, R\right) \prod_{i=1}^{R-1} \tfrac{1+z-i-s}{(1+z-i)} \end{aligned}$$

Note that if $k - 1 \geq s > z$ then $G_1 = 0$.

4.6.8 Deriavtion of (4.27)

Suppose that in the model of section (4.27) the representative firm competes for consumer attention with $n\phi - 1$ competitors. Then (4.3) - (4.5) give

$$E[q_k, R] = p\left(f, \bar{f}, n\phi, R\right) p\left(k \,|A\right) = p\left(f, \bar{f}, n\phi, R\right) \prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i} \tag{4.48}$$

Hence for $\bar{f} = f$ (3.21) implies

$$\frac{\partial E[q_k, R]}{\partial f} \cong \frac{n\phi - R}{(n\phi)^2 f} R \prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i}$$

This suggests to set

$$\sum_{k=1}^{n} \frac{\partial E\left[q_k, R\,|j \in I_k\right]}{\partial f} \cong \frac{n\phi - R}{(n\phi)^2 f} R \sum_{k=1}^{n} \prod_{i=1}^{R-1} \frac{1+n-k-i}{n-i} = \frac{n\phi - R}{(n\phi)^2 f} n$$

where the second equality follows from lemma 4.3 (see 4.6.1).

To illustrate the quality of the approximation I compare (4.27) to the true formulation. Using (4.22) we get

$$\begin{aligned} \tfrac{\partial E[q_k,R|j\in I_k]}{\partial f} &= \sum_{z=R}^{n-1}\sum_{s=0}^{k-1} \phi^z (1-\phi)^{n-1-z} \binom{k-1}{s} \binom{n-k}{z-s} p_1\left(f, \bar{f}, 1+z, R\right) \prod_{i=1}^{R-1} \tfrac{1+z-i-s}{(1+z-i)} \\ &= \tfrac{R}{f} \sum_{z=R}^{n-1}\sum_{s=0}^{k-1} \phi^z (1-\phi)^{n-1-z} \binom{k-1}{s} \binom{n-k}{z-s} \tfrac{1+z-R}{(1+z)^2} \prod_{i=1}^{R-1} \tfrac{1+z-i-s}{(1+z-i)} \end{aligned} \tag{4.49}$$

where the last line uses the ACF and (3.21). As expected the suggested approximation works numerically very well if n is large (compared to R) or ϕ is close to one. This is illustrated in figure 4.15. In the figure I compare (4.27) (the dashed line) to $\sum_{k=1}^{n} \frac{\partial E[q_k,R|j\in I_k]}{\partial f}$ where $\frac{\partial E[q_k,R|j\in I_k]}{\partial f}$ is given by (4.49). We see that the curves are close together and have a very similar shape. All simulations I have conducted show that if we use the approximation (4.27) in the equilibrium equations the comparative statics are always the same as if (4.49) were used instead.

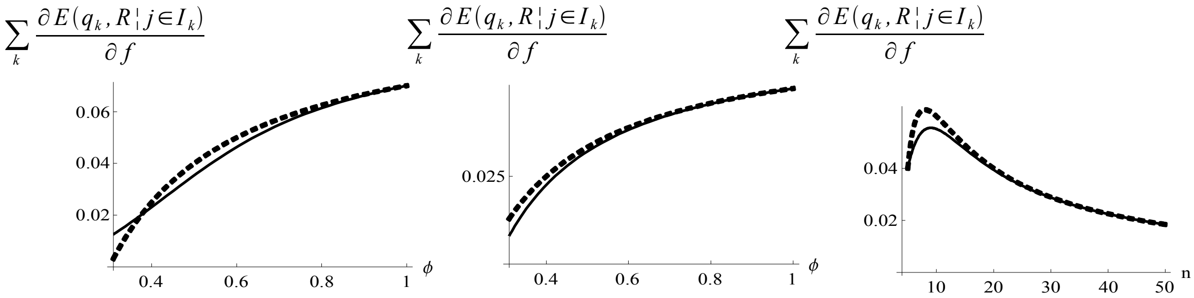

Figure 4.15: Quality of approximation (dashed) for $R = 3$, $n = 10$, $n = 30$ and $\phi = 1/2$

4.6.9 Proof of proposition 4.6

Note that $\lim_{\phi \to 0} \psi(\phi) = +\infty$ and $\lim_{\phi \to 1} \psi(\phi) = -\infty$. Further $\psi(\phi)$ is continuous on $\phi \in (0, 1)$ and $\psi'(\phi)$ exists unless $\phi = R/n$. For $\phi < R/n$ we have $\psi'(\phi) = -\frac{2t}{n^2\phi^3} - A''(\phi) < 0$. For $\phi > R/n$ we have $\psi'(\phi) < 0$ if $g(\phi) \equiv \left(1 - \frac{n\phi - R}{\eta} \frac{R}{n} \frac{A'(\phi)}{A(\phi)}\right)$ decreases in ϕ. But $g'(\phi) < 0$ holds if $(R - n\phi)(\varepsilon - \nu) < n\phi$. But as $(\nu - \varepsilon) \leq 1$ holds by assumption the condition is satisfied. This proves the existence of a unique interior ϕ that solves $\psi(\phi) = 0$. If $\psi(R/n) = t\Delta/R^2 - A'(R/n) > 0$ then because $\psi'(\phi) < 0$ for $\phi > R/n$ we must have $\tilde{\phi} > R/n$ at $\psi(\tilde{\phi}) = 0$.

■

I show that $\frac{\phi A'(\phi)}{A(\phi)} - \frac{\phi A''(\phi)}{A'(\phi)} < 1$ holds with the cost function $A(\phi) = \theta r \phi(m)$ based on (2.5). We have

$$A(\phi) = \theta r \frac{Ln\left(1 - \frac{(1-a)\phi}{r}\right)}{Ln(a)}$$

which requires $r > (1 - a)\phi$.Then differentiation yields

$$\Omega \equiv \frac{\phi A'(\phi)}{A(\phi)} - \frac{\phi A''(\phi)}{A'(\phi)} = \frac{(a-1)\phi\left(1 + Ln\left(1 - \frac{(1-a)\phi}{r}\right)\right)}{(r + (a-1)\phi)Ln\left(1 - \frac{(1-a)\phi}{r}\right)}$$

Then

$$\begin{aligned}
\Omega < 1 \quad &\Leftrightarrow \quad (a-1)\phi\left(1 + Ln\left(1 - \tfrac{(1-a)\phi}{r}\right)\right) > (r - (1-a)\phi)\, Ln\left(1 - \tfrac{(1-a)\phi}{r}\right) \\
&\Leftrightarrow \quad (a-1)\phi > r Ln\left(1 - \tfrac{(1-a)\phi}{r}\right) \\
&\Leftrightarrow \quad \tfrac{(a-1)\phi}{r} > Ln\left(1 + \tfrac{(a-1)\phi}{r}\right) \\
&\Leftrightarrow \quad x > Ln\,(1 + x)
\end{aligned}$$

where $x \equiv \frac{(a-1)\phi}{r} > -1$. But $x > Ln\,(1+x)$ is true for $x > -1$.

4.6.10 Proof of lemma 4.2

As before the average distance travelled by a consumer who consumes his k-th favourite variety is given by $\bar{w}_k = (2k-1)/4n$. Then the probability to consume the k-th favourite variety is given by $E\,[q_k, R] = \phi E\,[q_k, R\,|j \in I_k]$. Hence $\sum\limits_{k=1}^{n} \frac{2k-1}{4n} E[q_k, R]$ is average transportation distance of a consumer. The probability to be informed at least by one firm is given by $1-(1-\phi)^n$. Thus (4.33) is the average transportation cost of a consumer conditional on being informed of at least one variety.

■

4.6.11 Proof of proposition 4.7

For simplicity I set $E[q_k] \equiv E[q_k, \infty]$. Let $k^* \equiv 2+n-R$. Then for any $j \geq k^*$ we have $E[q_j] = E[q_j, R]$. To see this suppose the consumer is informed of his k-th best firm where $k \geq k^*$. Note that for any $k \geq k^*$ there are less than $R-2$ inferior firms. Hence even if the consumer ignores all inferior firms he will have at least one superior firm in his attention set and hence does not consume at the k-th best firm. Thus a consumer will only consume from his k-th best firm if he receives no information of a superior firm which is $E[q_j]$. Note that lemma 2.1 from chapter 2 implies

$$\sum_{k=1}^{k^*} E[q_k] = \sum_{k=1}^{k^*} E[q_k, R] \tag{4.50}$$

Next assume that

$$\begin{array}{c} \left(E_1^L, ..., E_w^L\right) < \left(E_1^{NL}, ..., E_w^{NL}\right) \\ \left(E_{w+1}^L, ..., E_{k^*}^L\right) \geq \left(E_{w+1}^{NL}, ..., E_{k^*}^{NL}\right) \end{array} \tag{4.51}$$

where $w < k^*$ and $E_k^L \equiv E[q_k, R]$ as well as $E_k^{NL} \equiv E[q_k]$. Thus we assume that the first w expectations under limited attention are strictly smaller than without limited attention. Next note that

$$\sum_{k=w+1}^{k^*} E_k^L = \sum_{k=1}^{k^*} E_k^{NL} - \sum_{k=1}^{w} E_k^L \tag{4.52}$$

The claim is that $T^L > T^{NL}$. To proof the claim note that it suffices to proof that

$$\sum_{k=1}^{k^*} (2k-1))\, E_k^L > \sum_{k=1}^{k^*} (2k-1))\, E_k^{NL} \tag{4.53}$$

Start with

$$\begin{aligned}
\sum_{k=1}^{k^*} (2k-1))\, E_k^L &= \sum_{k=1}^{w} (2k-1))\, E_k^L + \sum_{k=w+1}^{k^*} (2k-1))\, E_k^L \\
&= \sum_{k=1}^{w} (2k-1))\, E_k^L + (2w+1) \sum_{k=w+1}^{k^*} E_k^L + \sum_{k=w+2}^{k^*} 2\,(k-(1+w))\, E_k^L \\
&= (2w+1) \sum_{k=1}^{k^*} E_k^{NL} + \sum_{k=1}^{w} 2\,(k-(1+w))\, E_k^L + \sum_{k=w+2}^{k^*} 2\,(k-(1+w))\, E_k^L \\
&= (2w+1) \sum_{k=1}^{k^*} E_k^{NL} + \sum_{k=1}^{k^*} 2\,(k-(1+w))\, E_k^L
\end{aligned}$$

where I used (4.52) in the third line. Then (4.53) becomes

$$(2w+1) \sum_{k=1}^{k^*} E_k^{NL} + \sum_{k=1}^{k^*} 2\,(k-(1+w))\, E_k^L > \sum_{k=1}^{k^*} (2k-1)\, E_k^{NL}$$

which is equivalent to

$$\sum_{k=1}^{k^*} 2\,((1+w)-k)\, E_k^{NL} + \sum_{k=1}^{k^*} 2\,(k-(1+w))\, E_k^L > 0$$

and can be further reduced to

$$\sum_{k=1}^{k^*} ((1+w)-k)\, E_k^{NL} > \sum_{k=1}^{k^*} ((1+w)-k)\, E_k^L \tag{4.54}$$

Expand (4.54) and rearrange:

$$\sum_{k=1}^{w} ((1+w)-k)\, E_k^{NL} + \sum_{k=w+1}^{k^*} (k-(1+w))\, E_k^L > \sum_{k=1}^{w} ((1+w)-k)\, E_k^L + \sum_{k=w+1}^{k^*} (k-(1+w))\, E_k^{NL} \tag{4.55}$$

But (4.55) holds under (4.51). Thus we only need to show that (4.51) is satisfied. For $k < k^*$ and fixed R we have

$$\begin{aligned} E_k^{NL} - E_k^L &= \sum_{z=R}^{n-1} \phi^z (1-\phi)^{n-1-z} \binom{n-k}{z} \\ &- \sum_{z=R}^{n-1} \phi^z (1-\phi)^{n-1-z} \underbrace{\sum_{s=0}^{k-1} \binom{k-1}{s} \binom{n-k}{z-s} \xi(z,s)}_{\equiv A} \end{aligned} \tag{4.56}$$

where $\xi(z,s) \in [0,1]$. Hence

$$A(k) = \binom{n-k}{z} \xi(z,0) + \binom{k-1}{1} \binom{n-k}{z-1} \xi(z,1) + ... \tag{4.57}$$

For $k = 1$ we see from (4.56) that $E_k^{NL} - E_k^L > 0$ because $\xi(z,0) < 1$. I now show that there exists $\hat{k} \in [2, 2+n-R)$ such that

$$\binom{k-1}{1} \binom{n-\hat{k}}{z-1} \xi(z,1) > \binom{n-\hat{k}}{z} \tag{4.58}$$

holds for any $R \leq z \leq n-1$. In this case we must have $A(\hat{k}) > \binom{n-k}{z}$ and hence $E_j^{NL} - E_j^L < 0$ for all $\hat{k} \leq j < 2+n-R$. Using the definition of the binomial coefficient condition (4.58) can be simplified to

$$(k-1)\frac{R(1+z-R)}{(1+z)} > (n-k-z+1)$$

which is equivalent to

$$(n-z) < (k-1)\left(\frac{(1+z)(R+1)-R^2}{(1+z)}\right) \tag{4.59}$$

But setting $z = R$ makes the LHS of (4.59) as large as possible and also the RHS as small as possible. Hence

$$(n-R)\frac{1+R}{2R+1} + 1 < k \tag{4.60}$$

But as $(n-R)\frac{1+R}{2R+1} < n-R$ there must exist a $\hat{k} \in [2, n-R]$ such that (4.59) is true for any $R \leq z \leq n-1$. Hence (4.51) must be true.

$\blacksquare$

5

Differentiable Symmetric Games: Uniqueness and Stability

5.1 Introduction

In this chapter I develop a comparably simple approach to verify whether a differentiable symmetric game has a unique equilibrium and the equilibrium is symmetric. The main applications of the results in this chapter can be found in chapter 3 of this book but the approach to uniqueness presented here has a substantial stand-alone value.

My approach to uniqueness is different from the literature as I distinguish between multiple symmetric equilibria and asymmetric equilibria. By taking explicit advantage of the symmetry in the game I derive two criteria that, if both are satisfied, assert that only one equilibrium, the symmetric equilibrium, exists. One criterion rules out the possibility of multiple symmetric equilibria and the other criterion rules out the possibility of asymmetric equilibria.

Standard criteria of uniqueness such as the contraction, univalence or the index approach have the drawback that even in symmetric games they are not evaluated easily, notably when there are many players. The method presented in this chapter shows that the complexity of examining uniqueness in a general N-player game can be reduced to the case of a two-player game. Moreover, the index theorem, commonly accepted as the most general approach to uniqueness, is only applicable under the assumption of severe boundary conditions - a requirement that can be loosend as this chapter reveals.

This chapter is structured as follows. After building up the required notation and stating a sufficient condition for the existence of a symmetric equilibrium I derive and discuss the two criteria, that together assert uniqueness of the symmetric equilibrium in case of one- and two-dimensional strategy spaces[1]. As an interesting application I can discuss non-existence of asymmetric equilibria in case of two-dimensional supermodular symmetric games. To my knowledge this is the first contribution that provides a simple and intuitive characterisation of the scope of asymmetric equilibria in such games. I demonstrate the generality of my approach by showing that these two condition, when applied to the Cournot game, correspond exactly to the two general conditions of uniqueness that have been recognized by the literature. My approach enhances these conditions with a new interpretation. In section 5.4 I discuss the important relationship between stability of a symmetric equilibrium, uniqueness and comparative-static results. As was pointed out by Dixit there generally is a close and non-negligible link between stability and the comparative-static predictions of a game (Dixit (1986)). In symmetric games it turns out that the condition which rules out the existence of multiple symmetric equilibria also asserts stability under symmetric adjustments and crucially determines the sign of the comparative-statics. Finally, I derive a condition that asserts whether a symmetric equilibrium is a local contraction, which is simple to check, and yet more general than the usual requirement of a dominant diagonal.

5.2 Existence of symmetric equilibria

In this section I build up the notation and then derive a sufficient condition for the existence of a symmetric equilibrium.

5.2.1 Notation and assumptions

The notation follows Topkis (Topkis (1998)). Consider a non-cooperative game of $2 \leq N < \infty$ identical players. Players are indexed by a number $1, ..., N$ and all players have the same strategy space. Throughout the chapter strategies are defined to be pure strategies.

[1] I also illustrate that the method extends in a straightforward way to higher dimesnional games.

Let $x_g \equiv (x_{g1}, ..., x_{gk}) \in S(k)$ denote a feasible strategy of player g where $S(k) \subset \mathbb{R}_+^k$ denotes the strategy space of the player. Let $S \equiv S(k)$ be the product space $S = \times_{i=1}^k S_i$ where $S_i = \left[0, \bar{S}_i\right] \subset \mathbb{R}_+$ with $\bar{S}_i > 0$ for all $i = 1, ..., k$. Thus S is a nonempty, compact and convex subset of an Euclidian space. Further $x = (x_1, ..., x_N) \in S^N \subset \mathbb{R}_+^{kN}$ denotes a strategy profile and $S^N = S \times ... \times S$ also is compact and convex. For a player g and a given strategy profile x the vector $x_{-g} \in S^{N-1}$ denotes the vector of strategies of all $(N-1)$ players except player g. Sometimes I simply write $x = (x_g, x_{-g})$. The payoff function of player g is a real-valued and differentiable function $\Pi^g(x)$ defined on S^N. Following Dasgupta and Maskin symmetry means permutation-invariance of the payoff functions (Dasgupta and Maskin (1986)):

$$\Pi^g (x_1, ..., x_g, ..., x_N) = \Pi^{\sigma(g)} \left(x_{\sigma(1)}, ..., x_{\sigma(g)}, ..., x_{\sigma(N)}\right) \tag{5.1}$$

for $g = 1, ..., N$ where σ is a permutation of the set $\{1, ..., N\}$. This especially implies that

$$\Pi^g (x_1, ..., x_g, ..., x_N) = \Pi^g \left(x_{\mu(1)}, ..., x_g, ..., x_{\mu(N)}\right)$$

for $g = 1, ..., N$ where μ is a permutation of $\{1, ..., N\} \setminus \{g\}$. For simple reference I set

$$\Pi (x_g, x_{-g}) \equiv \Pi^g (x_1, ..., x_g, ..., x_N) \qquad g = 1, ..., N$$

The non-cooperative k-dimensional symmetric game is the triple $\left(N, S(k)^N, \Pi\right)$. Throughout this chapter I shall maintain the following assumption on Π:

Assumption 5.1. *For $g \in \{1, ..., N\}$ the payoff function $\Pi(x_g, x_{-g}) \in C^2\left(S^N, \mathbb{R}\right)$ satisfies* (5.1) *and is strictly quasiconcave in $x_g \in S$ for any $x_{-g} \in S^{(N-1)}$.*

The property of permutation-invariance implies[2]

$$\frac{\partial \Pi^g (x_1, ..., x_g, ..., x_N)}{\partial x_g} = \frac{\partial \Pi^g \left(x_{\mu(1)}, ..., x_g, ..., x_{\mu(N)}\right)}{\partial x_g}$$

[2]Note that the following expression is a vector derivative.

For simple reference I set

$$\Pi_i\left(x_g, x_{-g}\right) \equiv \frac{\partial \Pi^g\left(x_1, ..., x_g, ..., x_N\right)}{\partial x_{gi}} \qquad 1 \leq i \leq k, \quad 1 \leq g \leq N$$

In a non-cooperative game every player g solves

$$\max_{x_g \in S} \quad \Pi\left(x_g, x_{-g}\right) \tag{5.2}$$

I denote the best response function of player g by

$$\varphi^g : S^{N-1} \to S \qquad , x_{-g} \mapsto \varphi^g(x_{-g})$$

and use $\varphi(x_{-g}) \equiv \varphi^g(x_{-g})$ for simple reference. Similarly,

$$\varphi_i^g : S^{N-1} \to S_i \qquad , x_{-g} \mapsto \varphi_i^g(x_{-g})$$

is the i-th component $(1 \leq i \leq k)$ of φ^g and I set $\varphi_i(x_{-g}) \equiv \varphi_i^g(x_{-g})$.

To find a symmetric equilibrium in specific symmetric games usually a simplified approach, the symmetric opponents form approach (SOFA), is used (e.g. Dixit (1986), p. 116 or Grossman and Shapiro (1984) or Salop (1979)). The SOFA takes player $g = 1$ and sets $x_{-1} = \bar{x}_{-1} \equiv (\bar{x}, ..., \bar{x})$ where $\bar{x} \in S$ and $\tilde{\Pi}\left(x_1, \bar{x}\right) \equiv \Pi^1\left(x_1, \bar{x}_{-1}\right)$ and then solves

$$\max_{x_1 \in S} \quad \tilde{\Pi}\left(x_1, \bar{x}\right) \tag{5.3}$$

Let $\tilde{\varphi}(\bar{x}) \equiv \varphi(\bar{x}_{-1}) \in S$ denote the set of maximisiers of (5.3). A symmetric equilibrium of $\left(N, S^N, \Pi\right)$ then is a point $x^* \in S^N$ with $x_1^* = ... = x_N^*$ and $x_1^* \in \tilde{\varphi}(x_1^*)$.

5.2.2 An existence result

It is well-known that the set of all pure-strategy equilibrium points of a non-cooperative game is identical to the set of fixed points of the joint best response correspondence (see, e.g. Topkis (1998), p. 179). The following proposition is easy to prove.

Proposition 5.1. *Assumption 5.1 implies the following two facts for the symmetric game*

$\left(N, S^N, \Pi\right)$*:*

(i) There exists a continuous joint best-response function

$$\phi : S^N \to S^N \quad , x \mapsto \begin{pmatrix} \varphi(x_{-1}) \\ \vdots \\ \varphi(x_{-N}) \end{pmatrix} \tag{5.4}$$

(ii) $\left(N, S^N, \Pi\right)$ *has a symmetric equilibrium* $x^* \in S^N$

<u>Proof</u>: Appendix (5.6.2)

A question frequently asked is, when do only interior equilibria exist. Let

$$\nabla F\left(x\right) \equiv \begin{pmatrix} \nabla\Pi^1(x_1, ..., x_N) \\ \nabla\Pi^2(x_1, ..., x_N) \\ \vdots \\ \nabla\Pi^N(x_1, ..., x_N) \end{pmatrix} \tag{5.5}$$

denote the pseudogradient (a Nk vector; Rosen (1965)) of $\left(N, S^N, \Pi\right)$ at x where $\nabla\Pi^g\left(x_1, ..., x_N\right)$ for $g = 1, ..., N$ is the gradient (a k-vector) of the payoff function of player g with respect to x_g. Let $\hat{E} \equiv \left\{x \in S^N : \phi(x) = x\right\}$ denote the set of all fixed points of ϕ. The set E of all interior equilibria (which might be empty) is

$$E \equiv \left\{x \in Int(S^N) : \nabla F\left(x\right) = 0\right\} \tag{5.6}$$

where $Int(S^N)$ denotes the interior of S^N. Let $J(x)$ denote the Jacobian (a $Nk \times Nk$ matrix) of $\nabla F\left(x\right)$ with respect to x. The following assumption asserts that only interior equilibria exist.

Assumption 5.2 (Boundary condition)**.** *The gradient field*

$$\nabla F : S^N \to \mathbb{R}^{Nk} \quad , x \mapsto \nabla F(x) \tag{5.7}$$

points into the interior of S^N *at all boundary points.*

Proposition 5.2. *The following two properties characterise the set of equilibria of* $\left(N, S^N, \Pi\right)$

i) Under assumption 5.2 only interior equilibria exist: $\hat{E} = E$

ii) If all equilibrium points are regular, i.e. the Jacobian $J(x)$ *has* $Det(J(x)) \neq 0$ *for all* $x \in E$*, then* E *is a finite set.*

Proof: Appendix (5.6.3)

Intuitively, i) is true because if assumption 5.2 holds but there also were a boundary equilibrium this would contradict the Kuhn-Tucker necessary and sufficient conditions. ii) holds because the regularity condition implies that all equilibrium points are locally isolated and a compact and "discrete" set must be finite. Note that assumption 5.2 is sufficient (but not necessary) for ruling out boundary equilibria.

When dealing with asymmetric equilibria I will require the best response function $\varphi(x_{-g})$ to be differentiable at certain points. As the following corollary states, assumption 5.1 asserts that $\varphi(x_{-g})$ is differentiable in x_{-g} if $\varphi(x_{-g}) \in Int(S)$.

Corollary 5.1. *If* $\varphi(x_{-g}) \in Int(S)$ *then* $\varphi(x_{-g})$ *is differentiable at* x_{-g}*.*

Proof: Appendix (5.6.4)

5.3 Uniqueness

This section first summarises the main two approaches to uniqueness: the univalence approach and the index approach. Then I develop the condition which excludes asymmetric equilibrira in case of one-dimensional strategy spaces first for the case of two players and then for the case of N players. Afterwards, I extend the results to the case of two-dimensional strategy spaces and $N \geq 2$. Then I introduce the condition for the inexistence of multiple symmetric equilibria. The section concludes with an application of these criteria to the Cournot game and a brief discussion of super- and submodular games.

5.3.1 Review: criteria for uniqueness

In order to determine whether or not a game has a unique equilibrium the literature has come up with three general approaches:

i) the contraction approach

ii) the univalence approach

iii) the Poincare-Hopf index theorem approach

I will first provide a short summary of the univalence approach and the index approach in the context of symmetric games and ignore the restrictive[3] contraction approach. Note that generally nothing detains us from applying these methods to a symmetric game but these methods have in common that they rely on boundary conditions and usually are impractical in the context of particular applications as the matrices involved can be very large.

5.3.1.1 The Univalence Approach

The univalence approach developed by Gale and Nikaido (Gale and Nikaido (1965)) provides sufficient conditions under which a mapping is globally univalent (i.e. one-to-one). The following is a reformulation of the univalence theorem from Gale and Nikaido (p. 89) in the context of a symmetric game.

Theorem 5.1 (Gale-Nikaido). *If all principal minors of $(-J(x))$ are positive for all $x \in S^N$ then at most one interior equilibrium can exist and this equilibrium must be symmetric.*

The obvious shortcomings of the theorem are i) it says nothing about boundary equilibria, ii) it is impractical as determining the signs of all principal minors of a $Nk \times Nk$-matrix is very tedious and iii) it is quite easily violated in applications as the following example shows:[4] Let $k = 1$ and $N = 2$ and suppose that $\Pi^1(x_1, x_2) = x_1\sqrt{x_2} - 1/2x_1^2$ and $\Pi^2(x_1, x_2) = x_2\sqrt{x_1} - 1/2x_2^2$. Then we have

$$\nabla F(x) = \begin{pmatrix} \sqrt{x_2} - x_1 \\ \sqrt{x_1} - x_2 \end{pmatrix} \qquad S = [1/5, 2] \tag{5.8}$$

[3] It is straightforward to show that the contraction approach is a special case of the univalence approach (see Vives (1999)).

[4] Although Rosen's version includes boundary equilibria (Rosen (1965)), the requirements of the payoff-function Π are stronger (concavity), it involves checking the definitness of a large matrix and also fails to establish uniqueness in the example below.

Then

$$-J(x) = \begin{pmatrix} 1 & -\frac{1}{2\sqrt{x_2}} \\ -\frac{1}{2\sqrt{x_1}} & 1 \end{pmatrix}$$

and $|-J(x)| = 1 - \frac{1}{4\sqrt{x_1 x_2}}$ which is not positive for all $(x_1, x_2) \in [1/5, 2]^2$. Nevertheless, the symmetric equilibrium $(1, 1)$ obviously is the only equilibrium.

5.3.1.2 The index theorem approach

The index theorem declares that, under assumption 5.2, we only need information about the sign of $Det\,(-J(x))$ at critical points instead of investigating all principal minors of $-J(x)$. Suppose all equilibria points are regular and interior. The index of a zero of the vector field $\nabla F(x)$ is then defined by (see Vives (1999), p.48):

$$I(x) = \begin{cases} +1 & \text{if } Det(-J(x)) > 0 \\ -1 & \text{if } Det(-J(x)) < 0 \end{cases} \tag{5.9}$$

It is known that if the strategy space forms a rectangle in some Euclidian space then $\mathcal{I} \equiv \sum_{x \in E} I(x) = 1$ where $I(x)$ is the index of a zero (see Simsek et al. (2007)). Hence if the boundary condition (assumption 5.2) holds and all equilibria points are regular then if

$$\nabla F(x) = 0 \quad \Rightarrow \quad Det\,(-J(x)) > 0 \tag{5.10}$$

holds the symmetric equilibrium is unique. Note that in example (5.8) the index theorem obviously implies $(1, 1)$ to be the unique equilibrium.[5] The index approach may still be hard to verify in applications as the matrix involved can be very large. Moreover, in the case of boundary equilibria or non-regular equilibria[6] the (standard) index theorem is not applicable[7].

[5] The index approach is more general as univalence because condition (5.10) is implied if all principal minors of $-J(x)$ are found to be positive. Nevertheless, the Gale-Nikaido theorem is often invoked in applied work as it asserts the global invertibility of inverse demands (Vives (1999), p. 76).

[6] In such a case the zeroes of the gradient field are not locally isolated which violates a prerequisite of the index theorem (Vives, 1999, p. 362).

[7] Note that a generalised version of the index theorem has been proven (Simsek et al. (2007)), that can deal with boundary equilibria. However, this approach still requires to calculate the determinant of a large matrix, and further side conditions (non-degeneracy and complementarity) of what the authors term a generalised critical point must be verified separately.

In the next two sections I develop the two criteria that together assert uniqueness of the symmetric equilibrium. I begin by discussing the possibility of multiple symmetric equilibria using the SOFA and then move on to the possibility of asymmetric equilibria.

5.3.2 Multiplicity of symmetric equilibria

In this section, based on the SOFA, I use the index theorem to establish a criterion that rules out the possibility of multiple symmetric equilibria. The set E^s of interior symmetric equilibria is given by

$$E^s \equiv \left\{ (x_1, ..., x_1) \in S^N : \nabla\tilde{\Pi}\,(x_1) = 0 \right\}$$

where $\nabla\tilde{\Pi}\,(x_1)$ is the gradient of $\tilde{\Pi}(x_1, \bar{x})$ with respect to x_1, evaluated at $\bar{x} = x_1$ and

$$\nabla\tilde{\Pi}\,(x_1) : S \to \mathbb{R}^k \quad , x_1 \mapsto \nabla\tilde{\Pi}\,(x_1) \tag{5.11}$$

is a vector field over S. Let $\tilde{J}(x_1)$ denotes the Jacobian of $\nabla\tilde{\Pi}(x_1)$.

Assumption 5.3 (Symmetric boundary and regularity)**.** *Let* $\nabla\tilde{\Pi}$ *be the vector field as defined by* (5.11)*. Then*

i) $\nabla\tilde{\Pi}$ *points inwards on the boundary of* S

ii) $Det\left(-\tilde{J}(x_1)\right) \neq 0$ *if* $\nabla\tilde{\Pi}(x_1) = 0$

Proposition 5.3. *Suppose assumption 5.3 is satisfied. Then all symmetric equilibria are interior and their number is odd. Further, if and only if*

$$\nabla\tilde{\Pi}(x_1) = 0 \quad \Rightarrow \quad Det\left(-\tilde{J}(x_1)\right) > 0 \qquad x_1 \in Int(S) \tag{5.12}$$

then there is only one symmetric equilibrium.

Proof: Follows directly from the index theorem as $\nabla\tilde{\Pi} : S \to \mathbb{R}^k$, $x_1 \mapsto \nabla\tilde{\Pi}(x_1)$ defines a vector field on S.

■

Let $\tilde{\Pi}_i(x_1)$ be the i-th component of $\nabla\tilde{\Pi}$. The boundary condition in proposition 5.3 then means that $\tilde{\Pi}_i(x_{11},...,0_i,...,x_{1k}) > 0$ and $\tilde{\Pi}_i\left(x_{11},...,\bar{S}_i,...,x_{1k}\right) < 0$ for $1 \leq i \leq k$. Obviously, if assumption 5.2 is satisfied then also assumption 5.3 i) holds. But assumption 5.2 is unnecessarily restrictive[8] because we can restrict ourselves to $\nabla\tilde{\Pi}$ (rather than $\nabla\Pi^1$) which obviously reduces the complexity of the task. All points that violate the boundary condition in proposition 5.3 are candidates for symmetric boundary equilibria. If the boundary condition in proposition 5.3 fails to hold and it is known that interior symmetric equilibria exist we may apply the univalence approach to $\tilde{J}(x_1)$: if all principal minors of $-\tilde{J}(x_1)$ are found to be positive for all $x_1 \in S$ and an interior symmetric equilibrium exists, this is the only interior symmetric equilibrium of the game. As we will see in section 5.4.1 there is an intimate relationship between (symmetric) stability of symmetric equilibria and the multiplicity of such equilibria which makes the comparative statics problematic in case of multiple symmetric equilibria. Thus from this perspective it makes sense, as a modelling advice, to impose boundary conditions as stated by proposition 5.3 directly on the symmetric opponent form of the game.

5.3.3 Inexistence of asymmetric equilibria

In this section I develop a sufficient criterion that excludes asymmetric equilibria from the equilibrium set. I start with the simple case where $k = 1$ and then generalise the result to the case $k = 2$. Finally, I illustrate how to extend the criterion to $k \geq 3$.

5.3.3.1 The case $k = 1$

Suppose $k = 1$ and $N \geq 2$. Let $\varphi(x_2; X) \equiv \varphi^1(x_2; X)$ where $X \equiv (x_3, ..., x_N)$. Note that because of corollary 5.1 the partial derivative $\partial\varphi(x_2; X) \equiv \frac{\partial}{\partial x_2}\varphi^1(x_2; X)$ exists if $\varphi(x_2; X) \in (0, \bar{S})$.

Theorem 5.2. *If for all $x_2 \in (0, \bar{S})$ and any given $X \in S^{N-2}$ for which $\partial\varphi(x_2; X)$ exists we have that $\partial\varphi(x_2; X) > -1$ then no asymmetric equilibrium exists.*

Proof: Appendix (5.6.5)

[8] If assumption 5.2 is satisfied the index theorem can be used to obtain a counting rule that reveals more regularities of the equilibrium set E (see 5.6.1 in the appendix).

Note that the shape of $\varphi(x_2; X)$ is not restricted by theorem 5.2 up to the slope condition for those points $(x_2; X)$ which imply that $\varphi(x_2; X) \in (0, \bar{S})$. Hence, other than the index theorem, we do not require any boundary conditions to hold nor do we need to exclude the possibility of non-isolated asymmetric equilibria (i.e. theorem 5.2 does not require assumption 5.2). Moreover, to exclude asymmetric equilibria from the equilibrium set by theorem 5.2 we can focus on a two-player version of the game where the vector X of strategies of the other player enter as exogenous parameters and the slope condition in theorem 5.2 is much simpler to work with than the determinant of a $N \times N$-matrix. A straightforward application of the IFT gives the following sufficient condition for the inexistence of asymmetric equilibria:

Corollary 5.2. *If for all $x_1, x_2 \in (0, \bar{S})$ and any given $X \in S^{N-2}$ we have that*

$$\Pi_1\left(x_1, x_2; X\right) = 0 \quad \Rightarrow \quad \frac{\Pi_{12}\left(x_1, x_2; X\right)}{\Pi_{11}\left(x_1, x_2; X\right)} < 1 \tag{5.13}$$

then no asymmetric equilibrium exists.

Note that corollary 5.2 means that we can exclude asymmetric *boundary* equilibria by considering only *interior* points in a two-player game.

In the remainder of this section I provide the simple geometric intuition behind theorem 5.2. In essence, it is an application of the Mean Value Theorem and the idea is illustrated in figure 5.1. Suppose the point $A = (x_1^a, x_2^a)$ corresponds to an asymmetric equilibrium of the game, as depicted in the figure. By symmetry its reflection, the point $A' = (x_2^a, x_1^a)$, must then also be an asymmetric equilibrium. As is depicted in the figure the line that connects A and A' must have a slope of -1. The best response function of player one, $\varphi(x_2)$, remains in the interior of $S = \left[0, \bar{S}\right]$ which according to corollary 5.1 means that this function must be differentiable for any $x_2 \in (0, \bar{S})$. But then the Mean Value Theorem tells us that there must be a point $\tilde{x}_2 \in (x_2^a, x_1^a)$ such that we have $\partial\varphi(\tilde{x}_2) = -1$. Theorem 5.2 then states that if we cannot find such a point $(\tilde{x}_2)$ we may conclude that no asymmetric equilibrium exists.[9]

[9]The complete proof (see appendix) is complicated by the fact that $\varphi(x_2)$ may be on the boundary

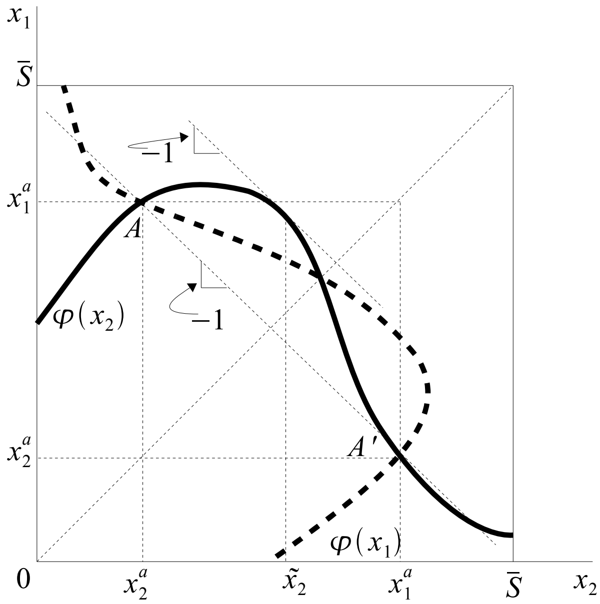

Figure 5.1: Theorem 5.2

5.3.3.2 The case $k = 2$ and $N \geq 2$

I now turn to the case where $k = 2$. To ease notation I introduce the four symbols $\alpha, \beta, \gamma, \delta$ which denote the partial derivatives of the best response function of $g = 1$ with respect to a strategy of player $g = 2$ for given $(x_{21}, x_{22}; X)$ where $X = (x_{31}, x_{32}, ..., x_{N1}, x_{N2})$.

$$\begin{aligned} \alpha &\equiv \frac{\partial \varphi_1(x_{21}, x_{22}; X)}{\partial x_{21}} \\ \beta &\equiv \frac{\partial \varphi_1(x_{21}, x_{22}; X)}{\partial x_{22}} \\ \gamma &\equiv \frac{\partial \varphi_2(x_{21}, x_{22}; X)}{\partial x_{21}} \\ \delta &\equiv \frac{\partial \varphi_2(x_{21}, x_{22}; X)}{\partial x_{22}} \end{aligned} \tag{5.14}$$

Theorem 5.3. *If for all $x_2 \in Int(S)$ and any given $X \in S^{N-2}$ for which the corresponding partial derivatives as defined by* (5.14) *exists we have that*

$$\begin{array}{ll} \alpha > -1 & \delta > -1 \\ \alpha + \delta + \alpha\delta - \beta\gamma > -1 & \end{array} \tag{5.15}$$

then no asymmetric equilibrium exists.

Proof: Appendix (5.6.6)

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and hence not differentiable everywhere.

Note that corollary 5.2 extends in a straightforward way to the case where $k = 2$. Theorem 5.3 is not only a useful tool in order to exclude asymmetric equilibria (interior or boundary) from the equilibrium set but it also permits to get some economically interesting insights in case of two-dimensional (supermodular) games (see section 5.3.5.2). In the appendix (see 5.6.6) I illustrate that theorem 5.3 can be extended in a straightforward way to the case where $k > 2$.

5.3.4 Discussion

If the conditions of proposition 5.3 and of theorem 5.2 or 5.3 (or the higher dimensional analogue) are satisfied, then the game only has one equilibrium, the interior symmetric equilibrium. Especially note that we only need to examine those boundary points that could be part of a symmetric boundary equilibrium and not the entire boundary - which is a clear practical advantage over the index theorem approach. Moreover, the slope condition of theorem 5.2 (or theorem 5.3) are simpler to evaluate than the index or the univalence condition as they depend only on the slope of the response function of an individual player with respect to a single other player and not on the Jacobian matrix of the entire game. Similarly, the condition of proposition 5.3 is simple to work with as it builds on the symmetric opponent form. Finally, as my approach to uniqueness finds two criteria which have a functional interpretation (one excludes asymmetric equilibria, one excludes multiple symmetric equilibria) we can also use these conditions to discuss economically interesting equilibrium features of a particular application.

5.3.5 Examples

To demonstrate the generality of the separation approach develop so far I show in the next section that its application to the Cournot model generates two conditions asserting uniqueness which are weaker than the main conditions of uniqueness as worked up by the literature. The implications of theorem 5.3 for two-dimensional (supermodular) games are discussed afterwards.

5.3.5.1 The Cournot game

The Cournot model is one of the most analysed economic models. Let

$$\Pi^1 = q_1 P(Q) - c(q_1) \qquad Q = \sum_{i=1}^{N} q_i$$

and $q_i \in [0, \bar{S}] \equiv S$. In accordance with assumption 5.1 I take $P(Q), c(q_1)$ to be twice continuously differentiable for $Q \in [0, N\bar{S}]$ and $q \in S$ with $P'(Q) < 0$ as well as $c'(q_1) > 0$ and $c''(q_1) \geq 0$ and assume strong quasiconcavity[10]. Then we have

$$\nabla\tilde{\Pi}(q) = P(Nq) + qP'(Nq) - c'(q) \tag{5.16}$$

The following assumption[11] rules out symmetric boundary optima[12]

Assumption 5.4 (Cournot boundary conditions)**.** *The following symmetric boundary conditions are imposed:*

i) $\nabla\tilde{\Pi}(0) = P(0) - c'(0) > 0$

ii) $\nabla\tilde{\Pi}(N\bar{S}) = P(N\bar{S}) + \bar{S}P'(N\bar{S}) - c'(\bar{S}) < 0$

Proposition 5.4 (Uniqueness in Cournot)**.** *Under assumption 5.4 and*

i) if for $q \in \left(0, \bar{S}\right)$ *we have that*

$$qP'(Nq) + P(Nq) - c'(q) = 0 \;\Rightarrow\; -\left(qNP''(Nq) + P'(Nq)(1+N) - c''(q)\right) > 0 \tag{5.17}$$

then only one symmetric equilibrium exists, and it is an interior equilibrium,

ii) if for $q_1 \in (0, \bar{S})$ *and any given* $Q \in \left(0, N\bar{S}\right)$ *we have that*

$$P(Q) + q_1P'(Q) - c'(q_1) = 0 \quad\Rightarrow\quad P'(Q) - c''(q_1) < 0 \tag{5.18}$$

[10] $P'(Q)q_1 + P(Q) - c'(q_1) = 0 \quad\Rightarrow\quad P''(Q)q_1 + 2P'(Q) - c''(q_1) < 0.$

[11] Note that this is the *only* boundary assumption required by the separation approach!

[12] If $\lim_{q\to 0} P(Nq) + qP'(Nq) > 0$ and $\lim_{q\to\infty} P(Nq) + qP'(Nq) \leq 0$ hold and the function $\xi(q) \equiv P(Nq) + qP'(Nq)$ is differentiable on $(0, \infty)$ we can always find $\underline{S} > 0, \bar{S} < \infty$ such that $\nabla\tilde{\Pi}(\underline{S}) > 0$ and $\nabla\tilde{\Pi}(\underline{S}) < 0$ holds and hence proposition 5.3 is applicable (as taking zero to be the lower bound of the strategy space obviously can be relaxed in proposition 5.3).

then no asymmetric equilibrium exists.

iii) If (5.17) *and* (5.18) *are both satisfied, then the Cournot game only has one equilibrium, the symmetric interior equilibrium*

Proof: Appendix (5.6.7)

Note that by ii) we may restrict ourselves to interior points of Q in order to exclude asymmetric (boundary) equilibria. Note that quasiconcavity does *not* imply that both conditions (5.17) and (5.18) need to hold but vice-versa if the two conditions hold then the payoff-function must be (srictly) quasiconcave at interior points.

How general are the conditions generated by the separation approach in case of the Cournot game? The condition of uniqueness[13] as worked up by Gaudet and Salant are widely accepted as very general conditions of uniqueness (Gaudet and Salant (1991), p. 401). Usually, the requirements that

$$\begin{aligned}(P' + q_1 P'') < 0 \\ c''(q_1) - P' > 0\end{aligned} \tag{5.19}$$

hold is invoked in applied work (see e.g. Vives (1999), p. 98). It is not hard to see that these conditions imply the Gaudet-Salant condition as well as the separation conditions in proposition 5.4. Moreover, as $P'(Q) - c''(q_1) < 0$ is an exogenous assumption of Gaudet-Salant (which is not necessarily required by the assumptions of the separation approach!), we see that precisely this assumption excludes the possibility of asymmetric equilibria in the symmetric game (on and off the boundary). The separation conditions obviously are a lot simpler to evaluate than the Salant-Gaudet condition. Finally, we can give the two inequalities in (5.19) a new interpretation as they rule out different types of equilibria. Note that $P''(Q)$ does not occur in (5.18). This means that, under quasiconcavity, the curvature of $P(Q)$ plays no role for determining whether there are asymmetric equilibria or not. However, the curvature of $P(Q)$ (the elasticity of $P'(Q)$ with respect to q_1) influences

[13] If

$$\sum_{i \in M(Q^E)} \frac{P'(Q^E) + q_i^E P''(Q^E)}{c''(q_i^E) - p'(Q^E)} < 1$$

holds, where $M(Q^E)$ is the set of players for which a candidate equilibrium q_i^E has $q_i' E(Q^E) > 0$, then only one equilibrium exists. Note that the Gaudet-Salant condition and the separation conditions depend on different (boundary) assumptions.

if there are multiple symmetric equilibria. In case of constant unit costs ($c'' = 0$) it is easy to see that the possibility of asymmetric equilibria in the symmetric Cournot game is rather special: a necessary condition for the existence of asymmetric equilibria in this game is that $-P' \leq 0$ is possible - which is contrary to the standard economic presumption of the model.

The fact that in the Cournot N-player game we may restrict ourselves to interior points of the aggregate quantity Q is not a coincidence but stems from the fact that the strategies of the other players affect the payoff-function as a sum:

Corollary 5.3. *If* $\Pi(x_1, ..., x_N) = \hat{\Pi}(x_1, Q)$, *where* $Q \equiv \sum_{j=1}^{N} x_j$, *and* $\hat{\Pi}_{11}(x_1, Q) + \hat{\Pi}_{12}(x_1, Q) < 0$ *holds for any* $x_1 \in (0, \bar{S})$ *and* $Q \in (0, N\bar{S})$ *with* $\hat{\Pi}_1(x_1, Q) + \hat{\Pi}_2(x_1, Q) = 0$, *then no asymmetric equilibrium exists.*

Proof:
In such a game we have $\Pi_1(x_1, ..., x_N) = \hat{\Pi}_1(x_1, Q) + \hat{\Pi}_2(x_1, Q)$, $\Pi_{12} = \hat{\Pi}_{12} + \hat{\Pi}_{22}$ and $\Pi_{11} = \hat{\Pi}_{11} + 2\hat{\Pi}_{12} + \hat{\Pi}_{22}$. Thus the claim follows directly from corollary 5.2.

■

Note that the Cournot condition ($P' - c'' < 0$) is just a special case of corollary 5.3.[14]

5.3.5.2 Super- and submodular games

If $k = 1$ and $N \geq 2$ a supermodular symmetric game has $\Pi_{12}(x_j, x_{-j}) \geq 0$ for all $x \in S^N$ whereas a submodular game has $\Pi_{12}(x_j, x_{-j}) \leq 0$. Thus with theorem 5.2 and proposition 5.3 we can easily replicate the result that supermodular games can never have any asymmetric equilibria[15] but can have multiple symmetric equilibria whereas submodular games never have multiple symmetric equilibria but there can be asymmetric equilibria.

It is known that the inexistence result of asymmetric equilibria in supermodular games does not extend to the case of multi-dimensional strategy spaces. That two-dimensional symmetric supermodular games may have asymmetric equilibria has been shown to hold

[14] Corollary 5.3 does *not* generally extend to the case of a general aggregative game (see Alos-Ferrer and Ania (2005), p.500, for a definition of such games).

[15] Note that example (5.8) is a supermodular game. Hence the failure of the univalence approach in this example is only driven by the fact that there could be multiple symmetric equilibria.

by means of (non-differentiable) examples (e.g. Amir et al. (2008), p. 311). Let $N = 2$. From (5.15) we see that a necessary condition for asymmetric equilibria to occur is that $\beta = \frac{\partial \varphi_1(x_{21},x_{22})}{\partial x_{22}}$ and $\gamma = \frac{\partial \varphi_2(x_{21},x_{22})}{\partial x_{21}}$ are large compared to $\alpha = \frac{\partial \varphi_1(x_{21},x_{22})}{\partial x_{21}}$ and $\delta = \frac{\partial \varphi_1(x_{21},x_{22})}{\partial x_{21}}$. This result is intuitive as β and γ refer to cross-over effects (e.g. how does a change of advertising intensity of my competitor affect my pricing decision) of the strategies whereas α and δ capture the direct effects (e.g. how does a price change of my competitor affect my pricing decision). Hence weak direct complementarity but strong cross-over complementarity is necessarily required to generate asymmetric equilibria. Suppose that two players play a supermodular price-advertising game as indicated above with the property that there is a very strong cross-over but a weak direct complementarity. Now assume that player two increases his price. This induces player one also to increase his price and advertising but, because of the strong cross-over complementarity, he increases advertising by more. This in turn means that player two continues to increase his price. Hence this type of "dynamic" reinforcement intuitively explains the economically interesting result why asymmetric "specialisation" equilibria in symmetric two-dimensional supermodular games may in fact occur.

However, if the game is supermodular ($k > 1$) and proposition 5.3 holds (such that there only is one symmetric equilibrium) then no asymmetric equilibrium can exist.[16] Hence for supermodular games proposition 5.3 is necessary and sufficient for uniqueness of the (symmetric interior) equilibrium.

In the case of a submodular game an evaluation of (5.12) shows that already for $k = 2$ such a game can possibly generate multiple symmetric equilibria.

Many economically interesting applications involve the case of two-dimensional strategy spaces. We can extract some modelling advice from theorem 5.3 to exclude asymmetric equilibria from the equilibrium set. Suppose that $\alpha, \delta > -1$ is known to hold. Especially, games were the strategies are nested in the sense that $\beta, \gamma \neq 0$ are interesting as such games cannot be solved independently for each strategy. The following properties then assert inexistence of asymmetric equilibria (on and off the boundary) of such games:

[16] This is a consequence of the fact that in supermodular symmetric games the extremal equilibria must always be symmetric; Vives (2005), p. 448.

- Either β or γ is zero at least at equilibrium candidates.[17]
- $\alpha, \delta > 0$ and $|\beta|, |\gamma| \leq 1$.
- $\beta \geq 0$ and $\gamma \leq 0$ (or vice-versa).

Finally, it is not hard to see by using the implicit function theorem that if

$$\begin{array}{l} \Pi_1 (x_{11}, x_{12}, x_{21}, x_{22}; X) = 0 \\ \Pi_2 (x_{11}, x_{12}, x_{21}, x_{22}; X) = 0 \end{array} \quad \Rightarrow \quad |\Pi_{ii}| > \sum_{j \neq i, j \leq 4} |\Pi_{ij}| \qquad i = 1, 2$$

holds for any $(x_1, x_2) \in Int(S^2)$ and any given $X \in S^{N-2}$ then $\alpha, \delta > -1$ as well as $1 + \alpha, 1 + \delta > \beta, \gamma$ and the game cannot have any asymmetric equilibria[18].

5.4 Stability

In this section I discuss the connection between (local) stability and symmetric equilibria. For example, Dastidar shows that in case of the symmetric Cournot game uniqueness of the equilibrium and its local stability are intimately related (Dastidar (2000), p.213). By introducing the concept of symmetric stability I show for one-dimensional games satisfying assumption 5.3 that local stability and the inexistence of multiple symmetric equilibria are in fact the same properties. However, for higher dimensional games stability is a stronger requirement than non-multiplicity of symmetric equilibria. Finally, by extensively exploiting the SOFA, I derive a condition that asserts whether an interior symmetric equilibrium of a k-dimensional game is a local contraction which is more general than the usual argument of local diagonal dominance.

5.4.1 Comparative statics and symmetric stability

As we consider static games, stability conditions are in any case "without foundation" (Dixit (1986), p. 107). Nevertheless, as Dixit highlights, there is a close link between the sign of a comparative-static prediction and the local stability of an equilibrium. It turns

[17] E.g. if the best-response subfunction $\varphi_1 (x_2, ..., x_N)$ depends only on the sub-strategy vector $(x_{j1})_{2 \leq j \leq N}$ then no asymmetric equilibrium can exist - independent of how φ_2 depends on x_{-1}.

[18] Hence if the Jacobian matrix $J(x)$ of $\nabla F(x)$ as defined by (5.5) has a dominant diagonal then condition (5.14) is trivially implied.

out that a modification of Dixit's concept of local myopic stability provides a fundamental link between stability under symmetric deviations and the (in)existence of multiple symmetric equilibria.

To illustrate the importance of such a consideration I reexamine the Cournot example from section 3.4.1. I assume that assumption 5.3 is satisfied and work with linear costs for simplicity: $c(q_1) = cq_1$. Let $q \in (0, \bar{S})$ denote a symmetric interior equilibrium. Then by the implicit function theorem:

$$q'(c) = \frac{1}{NP''(Q)q + (1+N)P'(Q)} \tag{5.20}$$

We immediately recognize $NP''(Q)q + (1+N)P'(Q)$ as the relevant term in (5.17) that determines whether or not there are multiple symmetric equilibria. Thus in the case of multiple symmetric equilibria we can get[19] $q'(c) > 0$ at a particular equilibrium point. In such a case we must have that strategic effects are of first-order importance, i.e. they reverse the direction of the comparative-statics as suggested by the direct effect. To see this consider an exogenous, symmetric change of unit costs c. Assuming that all competitors choose the same initial response $d\bar{q}$, the change of strategy of firm one, dq_1, can be determined by using the total differential of the first-order condition of the symmetric opponents form:

$$\underbrace{(N-1)\left(P''(Q)q_1 + P'(Q)\right)}_{A} d\bar{q} + \underbrace{\left(P''(Q)q_1 + 2P'(Q)\right)}_{B} dq_1 - dc = 0 \tag{5.21}$$

From section 3.4.1 we know that $P''(Q)q_1 + P'(Q) > 0$ is necessary for multiple interior symmetric equilibria to exist and $B < 0$ follows from strong quasiconcavity. The direct effect of the c-shock on q_1, $\frac{dq_1^D}{dc}$ can be found by holding the competitors' reaction fixed, i.e. $d\bar{q} = 0$. Then $\frac{dq_1^D}{dc} = 1/B$ and thus $\frac{dq_1^D}{dc} < 0$. The strategic effect is $\frac{dq_1}{d\bar{q}} = -A/B$ and corresponds to the slope of the best-response function $q_1(\bar{q})$ as defined by the FOC of the symmetric opponent form. From (5.21) we get that $\frac{dq_1}{dc} = \frac{1}{B} - \frac{A}{B}\frac{d\bar{q}}{dc}$. Hence in equilibrium

[19]Because proposition 5.3 is an index theorem result, in the case of multiple symmetric regular equilibria we must have $NP''(Q)q + (1+N)P'(Q) > 0$ for at least one equilibrium point q

$(dq_1 = d\bar{q} = dq)$:

$$\frac{dq}{dc} = \frac{1}{B}\frac{B}{A+B} = \frac{1}{A+B}$$

Thus we have $sign\left(\frac{dq}{dc}\right) = sign\left(\frac{dq_1^D}{dc}\right)$ if and only if $-B > A$ which is equivalent to the slope of $q_1(\bar{q})$ being less than one at an equilibrium point. Thus if only one symmetric equilibrium exists then we must have that $sign\left(\frac{dq}{dc}\right) = sign\left(\frac{dq_1^D}{dc}\right)$, i.e. strategic effects are of second-order importance (they influence only the magnitude of the change). Note that this condition is independent of whether there are asymmetric equilibria or not. Although (local) stability under symmetric adjustments is a restricted version of general myopic stability[20] this analysis uncovers that symmetric stability is a minimal consistency requirement for the comparative statics of a game and the inexistence of multiple symmetric equilibria asserts symmetric stability in the Cournot example.

I now establish the link between symmetric stability and multiplicity of symmetric equilibria more formally.

Following Dixit's approach to stability I define the myopic adjustment process by

$$\begin{aligned} &\dot{x}_{11} = s_1 \tfrac{\partial}{\partial x_{11}} \Pi\left(x_1, x_{-1}\right) \\ &\vdots \\ &\dot{x}_{1k} = s_k \tfrac{\partial}{\partial x_{1k}} \Pi\left(x_1, x_{-1}\right) \\ &\vdots \\ &\dot{x}_{Nk} = s_k \tfrac{\partial}{\partial x_{Nk}} \Pi\left(x_N, x_{-N}\right) \end{aligned} \tag{5.22}$$

where $s_1, ..., s_k > 0$ are arbitrary adjustment speeds (Dixit (1986)). A solution to (5.22) has the form $x(t) = (x_j(t))_{1\le j\le N}$, where $x_j(t) = (x_{j1}(t), ..., x_{jk}(t))$ is the vector trajectory of player j. The symmetric myopic adjustment process is a restricted version where we require the initial values $x(0)$ to be symmetric, i.e. $x_j(0) = x_i(0)$, $1 \le i, j \le N$. Under this restriction the time path $x_j(t)$ must be the same for all players and is the solution to

$$\begin{aligned} &\dot{x}_{11} = s_1 \tilde{\Pi}_1(x_1) \\ &\vdots \\ &\dot{x}_{1k} = s_k \tilde{\Pi}_k(x_1) \end{aligned} \tag{5.23}$$

[20] A symmetric equilibrium that is stable under symmetric adjustments need not be stable under general myopic adjustments.

Let $\hat{J}(x_1)$ denote the Jacobian corresponding to (5.23) and suppose that $x = (x_1, ..., x_1)$ is an interior symetric equilibrium. A sufficient condition for x to be symmetrically stable (i.e. stable under symmetric adjustments) is that $\hat{J}(x_1)$ only has negative eigenvalues (or eigenvalues with negative real parts). The following proposition reveals the relationship between symmetric stability and (non)-multiplicity of symmetric equilibria.

Proposition 5.5 (Symmetric stability and uniqueness)**.** *Suppose that assumption 5.3 is satisfied. Then the following statements hold:*

i) If for $k \geq 1$ multiple symmetric equilibria exist, then there must be unstable symmetric equilibria.

ii) If for $k = 2$ we have that

$$\nabla\tilde{\Pi}(x_1) = 0 \quad \Rightarrow \quad \tilde{\Pi}_{11}(x_1), \tilde{\Pi}_{22}(x_1) < 0, \; Det\left(\tilde{J}(x_1)\right) > 0 \tag{5.24}$$

then there only is one symmetric equilibrium and it is locally symmetrically stable.

iii) For $k = 1$ the following statements are equivalent:

(a) All symmetric equilibria are locally symmetrically stable

(b) There only exists one symmetric equilibrium

(c) Strategic effects are of second-order importance at symmetric equilibrium points:

$$sign\,(x_1'(c)) = sign\left(\frac{\partial}{\partial c}\tilde{\Pi}_1(x_1, c)\right)$$

Moreover, if x is a symmetrically stable equilibrium with $\Pi_{11}(x) < \Pi_{12}(x)$ then x is a stable equilibrium.

Proof: Appendix (5.6.8)

We see that symmetric stability and non-multiplicity of symmetric equilibria are the same properties of one-dimensional games. In such games, instability of an equilibrium can be caused only by dominant aggregate strategic effects. This close connection between non-multiplicity and stability generally breaks down in higher dimensions, although non-multiplicity is a necessary condition for symmetric stability (and hence also for general myopic stability). Hence stability is a stronger concept as uniqueness (see proposition 5.6 below) and not much can be said in general concerning the relationship between stability and the strength of strategic effects.

The following condition is sufficient for symmetric stability and uniqueness[21] for any $k \geq 1$:

Proposition 5.6 (Sufficient condition for symmetric stability)**.** *If the symmetric matrix* $\hat{M}(x_1) \equiv -\left(\hat{J}(x_1) + \hat{J}(x_1)^T\right)$ *is positive definite for any choice of* $s_1, ..., s_k > 0$ *then* $x = (x_1, ..., x_1)$ *is a locally symmetrically stable equilibrium. Further, if*

$$\nabla\tilde{\Pi}(x_1) = 0 \quad \Rightarrow \quad \hat{M}(x_1) \text{ positive definite}$$

then there only is one symmetric equilibrium and it is locally symmetrically stable.

Proof: Appendix 5.6.9

In the case of supermodular games the sufficient condition for stability becomes particularly simple:

Proposition 5.7 (Symmetric stability in supermodular games)**.** *If the game is supermodular and for* $\nabla\tilde{\Pi}(x_1) = 0$ *we have that all principal minors of* $-\tilde{J}(x_1)$ *are positive then* $x = (x_1, ..., x_1)$ *is a locally symmetrically stable equilibrium point.*

Proof: Appendix 5.6.10

5.4.2 Local contraction stability

In the literature on stability also discrete (or iterative) adjustment processes have been considered. I present a condition for an equilibrium to be a local contraction that depends only on the symmetric opponent form which should provide useful in applications and is more general than the requirement of diagonal dominance if $k > 1$.

[21] In the literature many different sufficient conditions for stability have been proposed. See Tang et al. (2007) for a modern survey on matrix stability.

Let $\tilde{\varphi}_i(\bar{x})$ denote the i-th component of the k-vector $\tilde{\varphi}(\bar{x})$ as introduced in section 5.2.1 and $\bar{x} = (\bar{x}_1, ..., \bar{x}_k) \in S$.

Theorem 5.4. *Suppose $x^* \in E^s$. If*

$$\max_{1 \leq i \leq k} \left\{ \sum_{j=1}^{k} \left| \frac{\partial \tilde{\varphi}_i(\bar{x}^*)}{\partial \bar{x}_j} \right| \right\} < 1 \tag{5.25}$$

then there exists $\varepsilon > 0$ such that $\phi(x)$ is a contraction on $V \equiv \bar{\mathbb{B}}(x, \varepsilon)$.

Proof: Appendix (5.6.11)

In words, condition (5.25) means that, for any subfunction i, the sum of the absolute values of the partial derivatives[22] with respect to the symmetric opponent k-vector $\bar{x}$ must be less than one. It should be noted that this approach to stability puts no restriction on the initial deviations of the players other than they must be in the contraction basin V (i.e. close to the equilibrium). In particular, initial deviations need not be symmetric. This is somewhat remarkable as condition (5.25) only makes use of the symmetric opponent form, i.e. we do not have to work with the entire payoff-function which considerably simplifies the analysis for higher dimensional games or games with complicated payoff-functions.

The usual sufficient condition for ϕ to be a contraction is that the Jacobian $J(x)$ has a dominant diagonal (e.g. Vives (1999), 1999, p. 47). For $k = 1$ diagonal dominance and (5.25) are the same requirement. If $k > 1$ then it is not hard to find an example (see 5.6.11 in the appendix) that shows condition (5.25) to be more general than the requirement of a dominant diagonal.

5.5 Conclusion

In this chapter I developed a comparably simple approach to uniqueness by deriving two criteria which together assert uniqueness of the interior symmetric equilibrium of a differentiable symmetric game with compact strategy space. One criterion excludes the possibility of multiple symmetric equilibria and one criterion excludes the possibility of asymmetric equilibria. I demonstrated the generality of my approach by applying it to the

[22]In applications, these derivatives can be calculated directly from the equilibrium condition $\nabla\tilde{\Pi}(x_1, \bar{x}) = 0$ by a direct application of the Implicit Function Theorem.

Cournot game. Further, I have used the criterion that deals with asymmetric equilibria to discuss the scope of asymmetric equilibria in two-dimensional symmetric supermodular games. The connection between comparative static predictions, multiplicity of symmetric equilibria and symmetric stability has been discussed. Finally, a condition for an interior symmetric equilibrium to be a local contraction was developed that is more general than the standard requirement of a dominant diagonal.

5.6 Appendix

5.6.1 A counting rule

Suppose assumption 5.2 is satisfied (so that only interior equilibria exist) and all equilibrium points are regular. If x^a is an asymmetric equilibrium then by the permutation-invariance property of ∇F for every possible alignment $\tilde{x}^a$ of the vectors $(x_1^a, ..., x_N^a)$ of x^a we must have that $\tilde{x}^a$ also is an asymmetric equilibrium. Thus it makes sense to introduce the notion of a distinct asymmetric equilibrium. Let $(x_1, x_2, ..., x_N), (x_1{'}, x_2{'}, ..., x_N') \in S^N$ denote two asymmetric equilibria. Then these are distinct asymmetric equilibria if they are not permutations of each other.

Proposition 5.8. *Suppose assumption 5.2 is satisfied and all equilibria points are regular. Let $\mathcal{I}^s$ denote the sum of the indices of all symmetric equilibria on the field of ∇F defined in* (5.7)*.*

1) If $\mathcal{I}^s = 1$, then if there are asymmetric equilibria

- *i) if $N \geq 2$ there must be more than one distinct asymmetric equilibrium*
- *ii) if $N = 2$ there must be an even number of distinct asymmetric equilibria*

2) If $\mathcal{I}^s \neq 1$ and $N \geq 2$ then asymmetric equilibria exist. If especially $N = 2$ then

- *i) if $\mathcal{I}^s = 3+4z$ for $z \in \mathbb{Z}$ then there must be an odd number of distinct asymmetric equilibria*
- *ii) if $\mathcal{I}^s = 5 + 4z$ for $z \in \mathbb{Z}\setminus\{-1\}$ then there must be an even number of distinct asymmetric equilibria*

Proof:

Because all equilibrium points are regular every equilibrium point has a well defined index which is either $+1$ or -1 (see (5.9)). Let $\omega \geq 1$ be the number of symmetric equilibria which is odd by proposition 5.3. Then $\mathcal{I}^s$ must be a number from $\{\pm 1, \pm 3, \pm 5, ..., \pm \omega\}$. Let $\mathcal{I}^a$ denote the index sum of all asymmetric equilibria. Then we must have $\mathcal{I}^s + \mathcal{I}^a = 1$ because S^N is a rectangle[23]. If $\mathcal{I}^s = 1$ but there are asymmetric equilibria then $\mathcal{I}^a = 0$ which requires at least two distinct asymmetric equilibria because all non-distinct asymmetric equilibria must have the same index. This proves 1) i). If $\mathcal{I}^s \neq 1$ we must have $\mathcal{I}^s \neq 0$ which proves that asymmetric equilibria must exist.

To see the rest set $N = 2$ and note that with $N = 2$ non-distinct asymmetric equilibria can only appear pairwise. Let n_1 denote the number of distinct asymmetric equilibria with index -1 and n_2 those with index $+1$. Then the index theorem implies that $n_2 - n_1 = \frac{1-\mathcal{I}^s}{2}$. Now, if $\mathcal{I}^s$ is a number $3+4z$ the RHS of this equation is an odd number which means that either n_1 or n_2 must be odd and the other number must be even or zero. Consequently, $n_1 + n_2$ must be odd. For $\mathcal{I}^s = 5+4z$ with $z \in \mathbb{Z} \setminus \{-1\}$ the RHS must be even and hence $n_1 + n_2$ must be even. Finally, if $\mathcal{I}^s = 1$ we must have $n_2 = n_1 = n$. If $n > 0$ then this implies $n_1 + n_2 = 2n$ which always is even.

■

5.6.2 Proof of proposition 5.1

Let $\varphi(x_{-g})$ denote the set of maximisers of player g from problem (5.2). Then $\varphi(x_{-g})$ is non-empty by the continuity of Π and the compactness of S . Moreover, $\varphi(x_{-g})$ is single-valued for each feasible x_{-g} because of the strict quasiconcavity of Π in x_g and the convexity of S. Finally, the continuity of Π implies the continuity of $\varphi(x_{-g})$ which, by symmetry, implies the continuity of ϕ. For the second claim note that $\tilde{\Pi}(x_1, \bar{x})$ is continuous and strongly quasiconcave in x_1 by the properties of Π^1. Hence the best-response function $\tilde{\varphi}(\bar{x}) \equiv \varphi(\bar{x}_{-1})$ exists, is continuous and a mapping from S to itself. The result then follows from the Brouwer FPT.

[23]A rectangle has an Euler characteristic of one which must correspond to the index sum of the vector field on the rectangle (see Simsek et al. (2007)).

■

5.6.3 Proof of proposition 5.2

First note that because Π^g is C^2 for $g = 1, ..., N$ the gradient field ∇F is well-defined. Because of symmetry we can concentrate on player $g = 1$ without loss of generality. Let $x = (x_1, x_{-1})$ and suppose $x_1 \in \partial S$, i.e. for at least one x_{1i} with $1 \leq i \leq k$ we have that $x_{1i} \in \{0, \bar{S}_i\}$. If $x_{1i} = 0$ then the boundary condition implies $\Pi_i(x_1, x_{-1}) > 0$ as otherwise the gradient field would not point inwards at x. Similarly, if $x_{1i} = \bar{S}_i$ then $\Pi_i(x_1, x_{-1}) < 0$. Suppose now that $x^* = (x_1^*, x_{-1}^*)$ is an equilibrium with $x_1^* \in \partial S$. Hence for at least one $1 \leq i \leq k$ we must have $x_{1i}^* \in \{0, \bar{S}_i\}$. If $x_{1i}^* = 0$ then by the Kuhn-Tucker necessary conditions we must have $\Pi_i\left(x_1^*, x_{-1}^*\right) \leq 0$ which contradicts the boundary condition. If $x_{1i}^* = \bar{S}_i$ then $\Pi_i\left(x_1^*, x_{-1}^*\right) \geq 0$, a contradiction. Hence only interior equilibria exist which proves i). For ii) note that E is bounded and because of the continuity of ϕ (see proposition 5.1 i)) also closed. Because $Det(J(x)) \neq 0$ at every $x \in E$ all equilibrium points are locally unique by the inverse function theorem. As E is compact and discrete (all points locally isolated) E must be finite.

■

5.6.4 Proof of corollary 5.1

Suppose that $\varphi(x_{-g}) \in Int(S)$. Then we must have $\nabla\Pi^g\left(x_1, ..., \varphi(x_{-g}), ..., x_N\right) = 0$. Strong quasiconcavity of Π^g and the fact that $\Pi^g \in C^2$ imply

$$\nabla\Pi^g\left(x_1, ..., \varphi(x_{-g}), ..., x_N\right) = 0 \quad \Rightarrow \quad Det\left(D_{x_g}^2\left(\Pi^g\left(x_1, ..., \varphi(x_{-g}), ..., x_N\right)\right)\right) \neq 0$$

where $D_{x_g}^2$ denotes the Jacobian of $\nabla\Pi^g$ with respect to x_g. Hence by the Implicit Function Theorem we may conclude that $\varphi(x_{-g})$ is differentiable at x_{-g}.

■

5.6.5 Proof of theorem 5.2

The proofs of theorems 5.2 and 5.3 require the following lemma.

Lemma 5.1. *Let $\psi \in C\left([0,1],[0,\bar{S}]\right)$ with $\psi(0) \neq \psi(1)$ and*

$$\psi(t) \in \left(0,\bar{S}\right) \quad \Rightarrow \quad \psi \text{ differentiable at } t$$

Then:

(i) *if $\psi(0) > \psi(1)$ $\exists t' \in (0,1)$ such that $-\psi'(t') \geq \psi(0) - \psi(1)$*

(ii) *if $\psi(0) < \psi(1)$ $\exists t'' \in (0,1)$ such that $\psi'(t'') \geq \psi(1) - \psi(0)$*

Proof: I first proove (i).
Suppose that $\psi(0) > \psi(1)$. Hence $\psi(0) > 0$ and $\psi(1) < \bar{S}$. Define $T \equiv \psi^{-1}\left(\left\{0,\bar{S}\right\}\right)$.
Case 1: $T = \emptyset$
As ψ is differentiable on $(0,1)$ and continuous on $[0,1]$ the MVT implies that $\exists t \in (0,1)$ such that $-\psi'(t) = \psi(0) - \psi(1)$.
Case 2: $T \neq \emptyset$
Obviously, T is bounded. Because $\left\{0,\bar{S}\right\}$ is closed and ψ is continuous T also is closed and thus a compact subset of $\mathbb{R}$. Hence the min and max of T exist and are denoted by $\underline{t}, \bar{t}$. The proof now is a case-by-case examination.

(a) $\psi(\underline{t}) = 0$. Then ψ is continuous on the perfect interval $[0,\underline{t}]$ and differentiable on $(0,\underline{t})$. Thus by the MVT $\exists t \in (0,\underline{t})$ such that

$$\psi'(t) = \frac{\psi(0) - \psi(\underline{t})}{-\underline{t}} \leq \frac{\psi(0) - \psi(1)}{-1}$$

(b) $\psi(\underline{t}) = \bar{S}$ and $\psi(\bar{t}) = \bar{S}$. Then ψ is continuous on the perfect interval $[\bar{t},1]$ and differentiable on $(\bar{t},1)$. By the MVT $\exists t \in (\bar{t},1)$ such that

$$\psi'(t) = \frac{\psi(\bar{t}) - \psi(1)}{\bar{t} - 1} \leq \frac{\psi(0) - \psi(1)}{-1}$$

(c) $\psi(\underline{t}) = \bar{S}$ and $\psi(\bar{t}) = 0$. Define $A \equiv \psi^{-1}\left(\left\{\bar{S}\right\}\right)$, which is a non-empty and compact set. Hence $\hat{t} = max\, A$ exists. Consider $B \equiv [\hat{t},1] \cap \psi^{-1}(\{0\})$, which also is non-empty and compact. Let $\check{t} = min\, B$. Hence ψ is continuous on the perfect interval

$[\hat{t}, \check{t}]$ and differentiable on $(\hat{t}, \check{t})$. Thus by the MVT $\exists t \in (\hat{t}, \check{t})$ such that

$$\psi'(t) = \frac{\psi(\hat{t}) - \psi(\check{t})}{\hat{t} - \check{t}} \leq \frac{\psi(0) - \psi(1)}{-1}$$

Hence (i) is true and (ii) follows from (i) by setting $\rho(z) \equiv \psi(1-t)$.

■

Proof of theorem 5.2

Step 1: $N = 2$

Suppose that x_1^a, x_2^a are asymmetric equilibria with the property that $\varphi(x_2^a) = x_1^a$ and $\varphi(x_1^a) = x_2^a$. Let $\psi(t) \equiv \varphi^1\left(x_1^a + t(x_2^a - x_1^a)\right)$ for $t \in [0,1]$. Then $\psi(0) = x_2^a$ and $\psi(1) = x_1^a$. Because $\psi(0) \neq \psi(1)$ lemma 5.1 together with proposition 5.1 (i) and corollary 5.1 asserts that there exists $t' \in (0,1)$ such that either $\psi'(t') \leq \psi(1) - \psi(0)$ or $\psi'(t') \geq \psi(1) - \psi(0)$. In the first case we get (by the chain rule) that $\partial\varphi \cdot (\psi(0) - \psi(1)) \leq \psi(1) - \psi(0)$ where $\psi(1) - \psi(0) < 0$, which implies $\partial\varphi(x_2) \leq -1$ for some $x_2 \in (0, \bar{S})$. In the second case we get $\partial\varphi \cdot (\psi(0) - \psi(1)) \geq \psi(1) - \psi(0)$ where $\psi(1) - \psi(0) > 0$. Consequently, again $\partial\varphi(x_2) \leq -1$ for some $x_2 \in (0, \bar{S})$.

Step 2: $N > 2$

Suppose $x^a \in S^N$ is an asymmetric equilibrium. Because of symmetry we can set $x^a = (x_1^a, x_2^a, x_3^a, ..., x_N^a)$ with $x_1^a \neq x_2^a$. Suppose players $g = 1$ and $g = 2$ play a parametrised two-player game where $X = (x_3, ..., x_N) \in S^{N-2}$ is an exogenous vector of parameters. If we choose $X = (x_3^a, ..., x_N^a)$ then (x_1^a, x_2^a) as well as (x_2^a, x_1^a) must be asymmetric equilibria of the parametrised two-player game. Thus, by step 1, if an asymmetric equilibrium exists we can always find $X \in S^{N-2}$ and $x_2 \in (0, \bar{S})$ such that $\partial\varphi(x_2; X) \leq -1$ - which is the contraposition of theorem 5.2.

■

5.6.6 Proof of theorem 5.3

The proof will make use of the following fact: If $r_1, r_2, s_1, s_2 \in \mathbb{R}$ with $r_1 \leq s_1 \leq 0$ and $r_2 \leq s_2 \leq 0$ then $r_1 r_2 \geq s_1 s_2$.

Let $N = 2$ and suppose an asymmetric equilibrium $x_1^a = (x_{11}^a, x_{12}^a) \in S$ exists. Then because of symmetry $x_2^a = (x_{21}^a, x_{22}^a)$ also is an asymmetric equilibrium where $\varphi(x_2^a) = x_1^a$ and $\varphi(x_1^a) = x_2^a$ and $x_1^a \neq x_2^a$. Define $\psi_1(t) \equiv \varphi_1^1(x_1^a + t(x_2^a - x_1^a))$ and $\psi_2(t) \equiv \varphi_2^1(x_1^a + t(x_2^a - x_1^a))$ for $t \in [0,1]$. Then $\psi_1(0) = \varphi_1^1(x_1^a)$, $\psi_1(1) = \varphi_1^1(x_2^a)$, $\psi_2(0) = \varphi_2^1(x_1^a)$ and $\psi_2(1) = \varphi_2^1(x_2^a)$. Moreover, $\psi_i \in C\left([0,1],[0,\bar{S}_i]\right)$ for $i = 1,2$ and also

$$\psi_i(t_i) \in \left(0, \bar{S}_i\right) \quad \Rightarrow \quad \psi_i \text{ differentiable at } t_i \text{ for i=1,2}$$

by corollary 5.1. Then, assuming that the corresponding partial derivative exist, by the chain rule we have for $i = 1$:

$$\psi_1{}'(t_1) = \alpha\left(x_1^a + t_1(x_2^a - x_1^a)\right)(\psi_1(0) - \psi_1(1)) + \beta\left(x_1^a + t_1(x_2^a - x_1^a)\right)(\psi_2(0) - \psi_2(1))$$

and for $i = 2$:

$$\psi_2{}'(t_2) = \gamma\left(x_1^a + t_2(x_2^a - x_1^a)\right)(\psi_1(0) - \psi_1(1)) + \delta\left(x_1^a + t_2(x_2^a - x_1^a)\right)(\psi_2(0) - \psi_2(1))$$

Case 1: $\varphi_i^1(x_1^a) = \varphi_i^1(x_2^a)$

Suppose that $\varphi_1^1(x_1^a) = \varphi_1^1(x_2^a)$. Hence $\psi_1(0) = \psi_1(1)$. Then $\varphi_2^1(x_1^a) \neq \varphi_2^1(x_2^a)$. According to lemma 5.1 there exists $t' \in (0,1)$ such that either $\psi_2{}'(t') \leq \psi_2(1) - \psi_2(0)$ or $\psi_2{}'(t') \geq \psi_2(1) - \psi_2(0)$. In the first case we get $\delta \cdot (\psi_2(0) - \psi_2(1)) \leq \psi_2(1) - \psi_2(0)$ where $\psi_2(1) - \psi_2(0) < 0$, which implies that $\exists (x_{21}, x_{22}) \in Int(S)$ such that $\delta(x_{21}, x_{22}) \leq -1$. In the second case we get $\delta \cdot (\psi_2(0) - \psi_2(1)) \geq \psi_2(1) - \psi_2(0)$ where $\psi_2(1) - \psi_2(0) > 0$. Consequently, again $\exists (x_{21}, x_{22}) \in Int(S)$ such that $\delta(x_{21}, x_{22}) \leq -1$. Finally, if instead $\varphi_1^2(x_1^a) = \varphi_1^2(x_2^a)$ then a similar argument gives that $\alpha \leq -1$ for some $(x_{21}, x_{22}) \in Int(S)$. From now on suppose that $\varphi_i^1(x_1^a) \neq \varphi_i^1(x_2^a)$ for $i = 1,2$ and define $m \equiv \frac{\psi_2(0) - \psi_2(1)}{\psi_1(0) - \psi_1(1)}$.

Case 2.1: $\varphi_i^1(x_1^a) > \varphi_i^1(x_2^a)$ for $i = 1,2$.

Then we have $\psi_i(0) > \psi_i(1)$ for $i = 1,2$ and $\exists t_i \in (0,1)$ such that $\psi_i{}'(t_i) \leq \psi_i(1) - \psi_i(0)$ which implies

$$\alpha + m\beta \leq -1$$
$$\gamma \tfrac{1}{m} + \delta \leq -1$$

Suppose that $\alpha, \delta \geq -1 \ \forall \ (x_{21}, x_{22}) \in Int(S)$ for which the corresponding derivative exists. Then

$$\begin{aligned} m\beta &\leq -(1+\alpha) \leq 0 \\ \gamma\tfrac{1}{m} &\leq -(1+\delta) \leq 0 \end{aligned}$$

and the above fact implies that $\exists (x_{21}, x_{22}), (x'_{21}, x'_{22}) \in Int(S)$ such that

$$\beta(x_{21}, x_{22})\gamma(x'_{21}, x'_{22}) \geq (1 + \alpha(x_{21}, x_{22}))(1 + \delta(x'_{21}, x'_{22}))$$

Case 2.2: $\varphi_i^1(x_1^a) < \varphi_i^1(x_2^a)$ for $i = 1, 2$.
Then we have $\psi_i(0) < \psi_i(1)$ for $i = 1, 2$ and $\exists t_i \in (0, 1)$ such that ${\psi_i}'(t_i) \geq \psi_i(1) - \psi_i(0)$ which implies

$$\begin{aligned} \alpha + m\beta &\leq -1 \\ \gamma\tfrac{1}{m} + \delta &\leq -1 \end{aligned}$$

and a similar argument as in case 2.1 shows that $\beta\gamma \geq (1 + \alpha)(1 + \delta)$ must be true if $\alpha, \delta \geq -1$.

Case 2.3: $\varphi_1^1(x_1^a) < \varphi_1^1(x_2^a)$ and $\varphi_2^1(x_1^a) > \varphi_2^1(x_2^a)$
Hence $\psi_1(0) < \psi_1(1)$ and $\psi_2(0) > \psi_2(1)$ and there exist $t_1, t_2 \in (0, 1)$ such that ${\psi_1}'(t_1) \geq \psi_1(1) - \psi_1(0)$ and ${\psi_2}'(t_2) \leq \psi_2(1) - \psi_2(0)$. Consequently, we get

$$\begin{aligned} m\beta &\leq -(1+\alpha) \leq 0 \\ \gamma\tfrac{1}{m} &\leq -(1+\delta) \leq 0 \end{aligned}$$

and a similar argument as in case 2.1 shows that $\beta\gamma \geq (1 + \alpha)(1 + \delta)$ must be true if $\alpha, \delta \geq -1$.

Case 2.4: $\varphi_1^1(x_1^a) > \varphi_1^1(x_2^a)$ and $\varphi_2^1(x_1^a) < \varphi_2^1(x_2^a)$ is treated similarly to case 2.3.

Thus if an asymmetric equilibrium exists we can always find interior points such that $\alpha \leq -1$, $\delta \leq -1$ or $(1 + \alpha)(1 + \delta) \geq \beta\gamma$ and the statement of theorem 5.3 simply is the contraposition of this result.

The proof is concluded by noting that for $N > 2$ if condition (5.15) holds for any given parameter vector $X \in S^{N-2}$ then, by the proof of theorem 5.2, there cannot be any asymmetric equilibrium in the game.

■

Remark: By defining

$$A \equiv \begin{pmatrix} 1+\frac{\partial\varphi_1^1}{\partial x_{21}} & \frac{\partial\varphi_1^1}{\partial x_{22}} \\ \frac{\partial\varphi_2^1}{\partial x_{21}} & 1+\frac{\partial\varphi_2^1}{\partial x_{22}} \end{pmatrix}$$

condition (5.15) of theorem 5.3 can be compactly written as the requirement that all principal minors of A be equal or greater than zero. Theorem 5.3 extends to the case where $k > 2$: Let

$$A = \begin{pmatrix} 1+\frac{\partial\varphi_1^1}{\partial x_{21}} & \frac{\partial\varphi_1^1}{\partial x_{22}} & \cdots & \frac{\partial\varphi_1^1}{\partial x_{2k}} \\ \frac{\partial\varphi_2^1}{\partial x_{21}} & 1+\frac{\partial\varphi_2^1}{\partial x_{22}} & \cdots & \frac{\partial\varphi_2^1}{\partial x_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi_k^1}{\partial x_{21}} & \cdots & \cdots & 1+\frac{\partial\varphi_k^1}{\partial x_{2k}} \end{pmatrix}$$

If all principal minors of A are greater or equal than zero no asymmteric equilibrium exists. To illustrate this suppose that $N = 2$ x_1^a and x_2^a are two asymmetric equilibria with $\varphi(x_1^a) = x_2^a$ and $\varphi(x_2^a) = x_1^a$ and $x_2^a \neq x_1^a$. For simplicity, assume that $\varphi(x_1^a + t(x_2^a - x_1^a)) \in Int(S)$ for $t \in (0,1)$. Then the mean value theorem asserts the existence of $x_2^1, ..., x_2^k \in Int(S)$ such that

$$\tilde{A}\begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_k \end{pmatrix} = -\begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_k \end{pmatrix} \tag{5.26}$$

where $\tilde{A}$ is a $k \times k$-matrix with $a_{ij} = \frac{\partial\varphi_i^1(x_2^i)}{\partial x_{2j}}$ and $\Delta_i \equiv x_{2i}^a - x_{1i}^a$. But (5.26) is equivalent to

$$\left(I+\tilde{A}\right)\begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_k \end{pmatrix} = A\begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_k \end{pmatrix} = 0 \tag{5.27}$$

Because x_1^a and x_2^a are asymmetric equilibria we must have $\Delta_i \neq 0$ for at least one i. Consequently, we must have $rank(A) < k$ in (5.27) and hence $Det(A) = 0$. Now suppose that $\Delta_k = 0$. Then (5.27) implies that

$$A_{k-1} \begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_{k-1} \end{pmatrix} = 0$$

where A_{k-1} is formed from A by cancelling the kth row and column. Hence $Det(A_{k-1}) = 0$, where $Det(A_{k-1})$ is a principal minor of order $k-1$ of A. Obviously, if $\Delta_j = 0$ for any $j = 1, ..., k$ then the corresponding principal minor of order $k-1$ of A must be zero. This argument may be continued to the case up to the case that $k-1$ of the k Δ_i's are zero and we find that this corresponds to the fact that at least one principal minor of A is equal to zero if an asymmetric equilibrium exists. Consequently, if we can show that all principal minors of A are positive we may conclude that no interior asymmetric equilibria nor asymmetric equilibria on the boundary with $\varphi(x_1^a + t(x_2^a - x_1^a)) \in Int(S)$ for $t \in (0,1)$ can exist. A somewhat tedious case-by-case examination as in the proof of theorem 5.3 shows that if $\varphi(x_1^a + t(x_2^a - x_1^a)) \notin Int(S)$ for $t \in (0,1)$ we can always find $x_2^1, ..., x_2^k \in Int(S)$ such that at least one principal minor of A is equal or less than zero.

5.6.7 Proof of proposition 5.4

(a) Applying (5.12) to (5.16) gives (5.17).

(b) From

$$\Pi_1 (q_1, q_2; q_3, ..., q_N) = P\left(q_1 + \sum_{j=2}^{n} q_j\right) + q_1 P'\left(q_1 + \sum_{j=2}^{n} q_j\right) - c'(q_1) = 0$$

we get

$$\frac{\Pi_{12}}{\Pi_{11}} = \frac{P'(Q) + q_1 P''(Q)}{2P'(Q) + q_1 P''(Q) - c''(q_1)}$$

and the claim follows from corollary 5.2.

(c) Obvious.

■

5.6.8 Proof of proposition 5.5

(a) Let $E_1 \equiv \left\{x_1 \in Int(S) : \nabla\tilde{\Pi}(x_1) = 0\right\}$ and suppose that E_1 is multi-valued. Then because of proposition 5.3 there must exist $x_1 \in E_1$ such that $Det(-\tilde{J}(x_1)) < 0$. Because $Det\left(-\hat{J}(x_1)\right) = s_1 \cdot ... \cdot s_k \cdot Det\left(-\tilde{J}(x_1)\right)$ we also have $Det\left(-\hat{J}(x_1)\right) < 0$. Let $\lambda_1, ..., \lambda_k$ denote the eigenvalues of $-\hat{J}(x_1)$. Then $\prod_{i=1}^{k} \lambda_k < 0$, which implies that there must exist at least one negative eigenvalue. Consequently, $\hat{J}(x_1)$ must have at least one positive eigenvalue which means that $x = (x_1, ..., x_1)$ is an unstable equilibrium.

(b) Inexistence of multiple symmetric equilibria is obvious. Let λ_1, λ_2 denote the eigenvalues of $\hat{J}(x_1)$. Then, because $s_1, s_2 > 0$, (5.24) implies:

$$\begin{aligned}\lambda_1 + \lambda_2 &= Trace\left(\hat{J}(x_1)\right) = s_1\tilde{\Pi}_{11} + s_2\tilde{\Pi}_{22} < 0\\ \lambda_1\lambda_2 &= Det\left(\hat{J}(x_1)\right) = s_1 s_2 Det\left(\tilde{J}(x_1)\right) > 0\end{aligned}$$

which implies that λ_1, λ_2 must either be negative or have negative real parts.

(c) (a) $\Leftrightarrow$ (b) follows as $Det\left(\hat{J}(x_1)\right), Det\left(\tilde{J}(x_1)\right) < 0$ holds iff $\tilde{\Pi}_{11}(x_1) < 0$ for $x_1 \in E_1$.

To see (b) $\Leftrightarrow$ (c) note that from $\tilde{\Pi}_1(x_1, \bar{x}, c) = 0$ we have that

$$x_1{}'(c) = -\frac{\frac{\partial}{\partial c}\tilde{\Pi}_1}{\Pi_{11}} - \frac{\Pi_{12}(N-1)}{\Pi_{11}}\bar{x}'(c)$$

where the first term is the direct effect and the second term the strategic effect. Because in equilibrium $(x_1'(c) = \bar{x}'(c))$

$$x_1{}'(c) = -\frac{\frac{\partial}{\partial c}\tilde{\Pi}_1}{\Pi_{11}} \frac{\Pi_{11}}{\Pi_{11} + (N-1)\Pi_{12}} = -\frac{\frac{\partial}{\partial c}\tilde{\Pi}_1}{\tilde{\Pi}_{11}}$$

we see that $sign\left(x_1{}'(c)\right) = sign\left(\frac{\partial}{\partial c}\tilde{\Pi}_1\right)$, i.e. strategic effects are of second-order importance, iff $\tilde{\Pi}_{11}(x_1) < 0$.

To see the last claim note that a symmetric equilibrium point x is locally stable under (general) myopic adjustments iff the symmetric $N \times N$-matrix

$$\mathrm{J(x)} = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{12} & & \Pi_{12} \\ \Pi_{12} & \Pi_{11} & \Pi_{12} & \cdots & \Pi_{12} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi_{12} & \Pi_{12} & \cdots & \Pi_{12} & \Pi_{11} \end{pmatrix}$$

is negative definite. This is the case iff

$$\Pi_{11}(x) < \Pi_{12}(x)$$

and

$$\Pi_{11}(x) + (N-1)\Pi_{12}(x) = \tilde{\Pi}_{11}(x_1) < 0$$

which holds by presupposition.

■

5.6.9 Proof of proposition 5.6

Suppose that $x_1 \in E_1$. If $\hat{M}(x_1)$ is positive definite $\forall\ s_1, ..., s_k > 0$ then i) all eigenvalues of $-\hat{J}(x_1)$ must be positive or have positive real part (see e.g. Tang et al. (2007)) and ii) $-\hat{J}(x_1)$ is a P-matrix (see e.g. Gale and Nikaido (1965), p. 84). But i) implies that $\hat{J}(x_1)$ only has negative eigenvalues or eigenvalues with negative real parts. Hence $x = (x_1, ..., x_1)$ must be a symmetrically stable equilibrium. Further, ii) implies that $Det(-\hat{J}(x_1)) > 0$ and the second claim follows from $Det(-\hat{J}(x_1)) = s_1 \cdot ... \cdot s_k Det(-\tilde{J}(x_1))$.

■

5.6.10 Proof of proposition 5.7

The supermodularity together with the positivity of the principal minors and $s_1, ..., s_k > 0$ imply that $-\hat{J}(x_1)$ is an M-Matrix (a matrix with non-positive off diagonal elements, positive diagonal elements and positive principal minors) for any choice of $s_1, ..., s_k > 0$.

But as all eigenvalues of an M-Matrix are known to be positive or have positive real part (see Tang et al. (2007)) the claim follows immediately.

■

5.6.11 Proof of theorem 5.4

The proof of theorem 5.4 requires the following two lemmata. The norm in use is $\|\cdot\| \equiv \|\cdot\|_\infty$. Let $\bar{\mathbb{B}}(x^*, \varepsilon)$ denote a closed ε-ball about x^*.

Lemma 5.2. *Suppose $x^* \in E^s$ and let ϕ be the joint best-response function* (5.4) *and $V \equiv \bar{\mathbb{B}}(x^*, \varepsilon) \subset S^N$. Then $\phi : V \to S^N$ is a contraction on V if $\varphi_i : \bar{U}_1 \to S_i$ for $\bar{U}_1 \equiv \bar{\mathbb{B}}\left(x^*_{-1}, \varepsilon\right) \subset S^{N-1}$ is a contraction on $\bar{U}_1$ for $i = 1, ..., k$.*

Proof:

Because of symmetry we need only regard player $g = 1$. Suppose that for every $1 \leq i \leq k$ φ_i is a contraction on $\bar{U}_1$ for a given $\varepsilon > 0$. Hence there exists $q_i \in [0, 1)$ with

$$x_{-1} \in \bar{U}_1 \quad \Rightarrow \quad \left|\varphi_i\left(x^*_{-1}\right) - \varphi_i\left(x_{-1}\right)\right| \leq q_i \left\|x^*_{-1} - x_{-1}\right\|$$

Let $q \equiv max\{q_i\}$ and $x_{-1} \in \bar{U}_1$. Then

$$\begin{aligned} \left\|\varphi(x^*_{-1}) - \varphi(x_{-1})\right\| &= \max_{1 \leq i \leq k} \left\{\left|\varphi_i(x^*_{-1}) - \varphi_i(x_{-1})\right|\right\} \\ &\leq q \left\|x^*_{-1} - x_{-1}\right\| \end{aligned}$$

Hence φ is a contraction on $\bar{U}_1$. By symmetry $\varphi(x_{-g})$ is a contraction on $\bar{U}_g \equiv \bar{\mathbb{B}}(x^*_{-g}, \varepsilon)$ for $g = 2, ..., N$. Suppose $x \in V$ and hence $x_{-g} \in \bar{U}_g$. Then

$$\begin{aligned} \left\|\phi(x^*) - \phi(x)\right\| &\leq q \left\|x^*_{-g} - x_{-g}\right\| \quad g = 1, ..., N \\ &\leq q \left\|x^* - x\right\| \end{aligned}$$

which shows that $\phi(x)$ is a contraction on V.

■

Lemma 5.3. *Suppose that $\sum_{j=1}^{k} \left|\frac{\partial \varphi_i(x^*_{-1})}{\partial x_{2j}}\right| < \frac{1}{N-1}$ and $x^* \in E^s$. Then there exists a closed ball $\bar{U}_i \subset S^{N-1}$ about (x^*_{-1}) such that φ_i^1 is a contraction on $\bar{U}_i$.*

Proof:

Define $\sigma_i(x_{-1}) \equiv \sum\limits_{j=1}^{k} \left|\frac{\partial\varphi_i(x_{-1})}{\partial x_{2j}}\right|$ and $q \equiv (N-1)\sigma_i(x^*_{-1})$. Then $q \in [0,1)$. As $\sigma_i(x_{-1})$ is continuous on S^{N-1} we have that for all $\varepsilon \in (0, \frac{1-q}{N-1}]$ there exists a $\delta(\varepsilon) > 0$ such that

$$\left\|x^*_{-1} - x_{-1}\right\| \leq \delta \Rightarrow \left|\sigma_i(x^*_{-1}) - \sigma_i(x_{-1})\right| < \varepsilon$$

Hence if $\sigma_i(x^*_{-1}) > \sigma_i(x_{-1})$ then $\sigma_i(x_{-1}) < \frac{q}{N-1} < \frac{1}{N-1}$. If $\sigma_i(x^*_{-1}) \leq \sigma_i(x_{-1})$ then

$$\sigma_i(x_{-1}) < \varepsilon + \sigma_i(x^*_{-1}) = \varepsilon + \frac{q}{N-1} \leq \frac{1}{N-1}$$

Let $\delta \equiv \delta\left(\frac{1-q}{N-1}\right)$ and $\bar{U}_i \equiv \bar{\mathbb{B}}\left((x^*_{-1}), \delta\right)$. Hence

$$x_{-1} \in \bar{U}_i \quad \Rightarrow \quad \sigma_i(x_{-1}) < \frac{1}{N-1}$$

Because of symmetry we have

$$x_{-1} \in \bar{U}_i \quad \Rightarrow \quad \sum_{j=1}^{k} \left|\frac{\partial\varphi_i(x_{-1})}{\partial x_{nj}}\right| < \frac{1}{N-1} \quad \forall n = 2, ..., N \tag{5.28}$$

By the mean value theorem:

$$|\varphi_i(x_{-1}{}') - \varphi_i(x_{-1})| \leq \|x_{-1}{}' - x_{-1}\| \sup_{\substack{0\leq t\leq 1 \\ \|\tilde{x}_{-1}\|=1}} \left|\nabla\varphi_i\left(x_{-1}{}' + t\left(x_{-1} - x_{-1}{}'\right)\right) \cdot (\tilde{x}_{-1})^T\right|$$

where $x_{-1}, x'_{-1} \in \bar{U}_i$ imply that also $x_{-1} + t\left(x'_{-1} - x_{-1}\right) \in \bar{U}_i$ for $t \in [0,1]$ as the closed ball $\bar{U}_i$ is a convex set. Let $x(t) \equiv x_{-1}{}' + t(x_{-1} - x_{-1}{}')$. Then

$$\begin{aligned}
Q &\equiv \sup_{\substack{0\leq t\leq 1 \\ \|\tilde{x}_{-1}\|=1}} |\nabla\varphi_i(x(t)) \cdot (\tilde{x}_{-1})| \\
&= \sup_{\substack{0\leq t\leq 1 \\ \|\tilde{x}_{-1}\|=1}} \left|\tfrac{\partial\varphi_i}{\partial x_{21}}(x(t))\tilde{x}_{21} + \tfrac{\partial\varphi_i}{\partial x_{22}}(x(t))\tilde{x}_{22} + ... + \tfrac{\partial\varphi_i}{\partial x_{Nk}}(x(t))\tilde{x}_{Nk}\right| \\
&= \sup_{0\leq t\leq 1} \left\{\left|\tfrac{\partial\varphi_i}{\partial x_{21}}(x(t))\right| + ... + \left|\tfrac{\partial\varphi_i}{\partial x_{2k}}(x(t))\right| + ... + \left|\tfrac{\partial\varphi_i}{\partial x_{N1}}(x(t))\right| + ... + \left|\tfrac{\partial\varphi_i}{\partial x_{Nk}}(x(t))\right|\right\} \\
&= \left|\tfrac{\partial\varphi_i}{\partial x_{21}}(t^*)\right| + ... + \left|\tfrac{\partial\varphi_i}{\partial x_{2k}}(x(t^*))\right| + ... + \left|\tfrac{\partial\varphi_i}{\partial x_{N1}}(x(t^*))\right| + ... + \left|\tfrac{\partial\varphi_i}{\partial x_{Nk}}(x(t^*))\right|
\end{aligned}$$

where the last step follows from the continuity of the partial derivatives and $t \in [0,1]$.

But (5.28) then implies

$$Q < \frac{N-1}{N-1} = 1$$

As a consequence

$$|\varphi_i(x_{-1}{}') - \varphi_i(x_{-1})| \leq Q \, \|x_{-1}{}' - x_{-1}\|$$

for all $x_{-1}, x'_{-1} \in \bar{U}_i$ and $Q \in [0,1)$. Hence φ_i^1 is a contraction on $\bar{U}_i$.

■

<u>Proof of theorem 5.4</u>:

Because of permutation-symmetry it does not matter which player marginally changes his strategy at a symmetric equilibrium. Then for $j = 1, ..., k$

$$\left|\frac{\partial \tilde{\varphi}_i(\bar{x}^*)}{\partial \bar{x}_j}\right| = (N-1)\left|\frac{\partial \varphi_i(x^*_{-1})}{\partial x_{2j}}\right|$$

Hence

$$\sum_{j=1}^{k}\left|\frac{\partial \tilde{\varphi}_i(\bar{x}^*)}{\partial \bar{x}_j}\right| = (N-1)\sum_{j=1}^{k}\left|\frac{\partial \varphi_i(x^*_{-1})}{\partial x_{2j}}\right|$$

and thus by presupposition

$$\sum_{j=1}^{k}\left|\frac{\partial \varphi_i(x^*_{-1})}{\partial x_{2j}}\right| < \frac{1}{N-1}$$

By lemma 5.3 there exists $\varepsilon > 0$ such that φ_i^1 is a contraction on $\bar{U}_i \equiv \bar{\mathbb{B}}_i(x^*_{-1}, \varepsilon_i)$. By the proof of lemma 5.2 $\varphi(x_{-1})$ is a contraction on $\bar{U} \equiv \bar{\mathbb{B}}(x^*_{-1}, \varepsilon)$ where $\varepsilon \equiv min\{\varepsilon_i\}$. But then by lemma 5.2 ϕ is a contraction on $V \equiv \bar{\mathbb{B}}(x^*, \varepsilon)$.

■

I now present a numerical example that violates the condition of a dominant diagonal but satisfies the contraction condition (5.25). Suppose $k = N = 2$ and x^* is an interior symmetric equilibrium. Let the Jacobian $J(x^*)$ be given by

$$J(x^*) = \begin{pmatrix} \Pi^1_{11} & \Pi^1_{12} & \Pi^1_{13} & \Pi^1_{14} \\ \Pi^1_{21} & \Pi^1_{22} & \Pi^1_{23} & \Pi^1_{24} \\ \Pi^2_{31} & \Pi^2_{32} & \Pi^2_{33} & \Pi^2_{34} \\ \Pi^2_{41} & \Pi^2_{42} & \Pi^2_{43} & \Pi^2_{44} \end{pmatrix} = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/2 & -1 & 1/8 & 1/8 \\ 0 & 1/2 & -1 & 1/2 \\ 1/8 & 1/8 & 1/2 & -1 \end{pmatrix}$$

As can be seen from the first row diagonal dominance fails to hold. However, using the implicit function theorem the partial derivatives are $\frac{\partial}{\partial x_{21}}\varphi_1(x_{21}, x_{22}) = 9/12$, $\frac{\partial}{\partial x_{22}}\varphi_1(x_{21}, x_{22}) = 1/12$, $\frac{\partial}{\partial x_{21}}\varphi_2(x_{21}, x_{22}) = 1/2$ and $\frac{\partial}{\partial x_{22}}\varphi_2(x_{21}, x_{22}) = 1/6$. Hence (5.25) is satisfied.

6

Summary

6.1 Summary

This thesis has integrated two stylized facts from marketing and perceptual psychology - the fact that people only consider a subset of all available alternatives and the fact that the salience of an alternative matters for its chance of perception - into oligopolistic models of competition. The basic implication of limited attention - the first fact - is that perceived and not effective market structure matters. I show that in an attention economy equilibrium prices generally depend on the intensity of competition between the firms which e.g. is determined by the degree of substitutability but also on the psychological attention threshold. This leaves firms with considerably more market power and implies that higher equilibrium prices can be maintained than suggested by conventional economic theory.

The second fact - the possibility to influence the chance of perception - implies the existence of a further type of competition - attention competition - that interacts with conventional economic competition. The fact that paid search has become the dominant part of online advertising expenditure clearly provides evidence for the importance of this type of competition in a digital economy. The theory of informative advertising recognises limited consumer information - because of scarce information - as a major source of inefficiency and market power. My contribution shows that it is of considerable importance for policy implications whether limited consumer information is caused by scarce information or by superabundant information and scarce attention. Depending on the intensity of price competition and the intensity of the competition for scarce attention firms may be able to transform the increased revenues into higher profits which under free entry implies larger markets. The Salop model provides an example where the revenue effect dominates which results in a higher equilibrium degree of diversity under limited attention. This is especially worrisome as under limited attention the conventional negative relationship between transportation costs and diversity is inverted.

6.2 Outlook

I hope of successfully having convinced the reader that the joint examination of psychological limitations on perception and economic competition is essential to understand and discuss many contemporaneous economic phenomena. It is also clear that this thesis is far from answering many intriguing questions that may be evoked in an attentive reader's mind. In this spirit I now provide a very brief description of two research questions which in my opinion deserve high attention.

As this thesis is of a theoretical nature maybe the most central concern is to find direct and indisputable empirical evidence that certify the validity of my claims. As Mondria et al. put it "empirical evaluation [...] of attention allocation models remains a challenge" (Mondria et al. (2010), p. 85). Fortunately, there exists a way to achieve such a goal by conducting a laboratory experiment. The main advantage of the laboratory over field data is the possibility to directly measure how many alternatives are perceived by participants (e.g. by recording their on-screen clicks) and also how much money information senders invest if they can influence their chance of perception and how their investment is related to the price they set if they are simultaneously engaged in some type of price-attention competition. I believe it is possible to construct a direct experimental test of the model in the third chapter of this thesis.

The second question is concerned with the assumption of separability imposed in chapter three. While this assumption is analytically very convenient as it generates a highly tractable structure there are cases where it seems inappropriate. An important example is the case of online pricing filters. Such websites draw together price information of different brands and list these informations in ascending order of their prices. Hence, different from the Google example, if such a filter is used by consumers then the on-screen ranking cannot be influenced by other means than the price itself. In my view this issue still can be addressed appropriately by the basic approach of chapter three by recognising that under limited attention the price must play a double role. On one hand, the price determines sales and revenues given the attention set of the consumer in the traditional way: lowering the price shifts up demand. On the other hand, given that people focus especially on the upper part of the list firms have an additional incentive to lower their price as this increases their chance of entering the upper part. Formally, this means that the model of chapter three is still applicable by letting attention effort f be a function of the price: $f = f(y)$. At first glance it appears that, propelled by their double role, prices should quickly converge to marginal costs as market power collapses. However, if the model is extended by a further dimension, e.g. quality, which is *not* measured by the pricing filter a quite different picture may be obtained. In such a case prices still decline but at the same time quality also drops because costs otherwise could not be covered. It turns out that in equilibrium such a price filter may imply a lower equilibrium effective quality than if the firms had to choose f directly. At an abstract level this example can illustrate the dangers of complacency about the preselection of information by digital filters. If such filters cannot identify all relevant criteria with respect to a specific request then blind reliance on the results may lead to bad conclusions. The inability of banks to understand their systemic risks or the deficient system used to identify potential terrorists by the U.S. government are two examples.

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